A_∞-structures in Lagrangian Floer Cohomology and String Field Theory

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A_∞-structures in Lagrangian Floer Cohomology and String Field Theory

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Chapter 1

Introduction

The whole story starts in the years 1988/´89 by Floer´s proof of the Arnold conjecture ([A]) for monotone symplectic manifolds ([FloeI], [FloeII]). By defining a boundary operator that counts orbits (*Floer trajectories*) connecting nondegenerate 1-periodic solutions of

$$\dot{x}\ (t) = X_{H_t}(x(t)) \tag{1.1}$$

for X_{H_t} being a family of time dependent Hamiltonian vector fields, he developed a new type of homology theory (*Floer homology*). By means of it he could proof that the number these 1-periodic solutions (in one-to-one correspondence with fixed points of time-1-symplectomorphisms on M) is bounded from below by the sum of the Betti numbers of M. Remark that this is a much stronger result than just to estimate it from below by the Euler characteristic (i.e. the alternating sum of the Betti numbers), achieved by using the Lefschetz fixed point theorem. This astonishing result is much stronger than topological examinations could provide and therefore it enforced the viewpoint that symplectic topology (large-scale perception due to the nonsqueezing theorem) is more precisely redenoted as symplectic geometry.
H. Hofer, D. Salamon, E. Zehnder any many others further worked out details of the proofs and therefore helped to develop the theory in order to achieve more generality for the requirements posed on M. The stated conjecture could be proven step by step for the semipositive case (e.g. [HoSa]) and finally for general compact symplectic manifolds (e.g. [FO]). To keep this text in a seizable manner we can neither discuss this process nor describe the pathbreaking results in the following. The interested reader is referred to the stated papers above or e.g. the lecture notes of D. Salamon [Sa] which provide a nice description of the conceptual buildup of this theory.

A. Floer proposed ([FloeI]) to apply his newly developed homology theory to face intersection issues of Lagrangian submanifolds

$$L_0^n, L_1^n \in (M^{2n}, \omega) \quad (\text{that is } \ 2 \cdot \dim\, L_i = \dim\, M,\ \omega|_{L_i} \equiv 0) \tag{1.2}$$

and tried to derive a similar lower bound estimate for the number of intersection points

$$\#\{p \in L_0 \cap L_1\}\ . \tag{1.3}$$

Such (non-)displaceabiltity questions are a major challenge in symplectic geometry. Compare for example the, conceptual quite different, ansatz of M. Entov and L. Polterovich in [EP]. The authors derived the concept of *quasi-states* (functionals on $C^0(M,\mathbb{R})$) in order to achieve more insight here.
The present text though focuses on and tries to extent the ideas originated by A. Floer. He defined a chain complex (over $\mathbb{Z}_2$) generated by intersection points

$$p \in L_0 \cap L_1 \ . \tag{1.4}$$

For the boundary operation

$$\partial p = \sum \langle \partial p, q \rangle q \tag{1.5}$$

one counts the number $\langle \partial p, q \rangle$ (mod $\mathbb{Z}_2$) of pseudo-holomorphic curves ($\frac{du}{ds} + J\frac{du}{dt} = 0$)

$$u : \mathbb{R} \times [0,1] \to M \tag{1.6}$$

attaching L_i, precisely speaking those that satisfy the boundary conditions (see figure 6.1)

$$\begin{aligned} \lim_{t\to-\infty} u(t,s) = p \quad &, \quad \lim_{t\to\infty} u(t,s) = q \\ u(t,s=0) \in L_0 \quad &, \quad u(t,s=1) \in L_1 . \end{aligned} \tag{1.7}$$

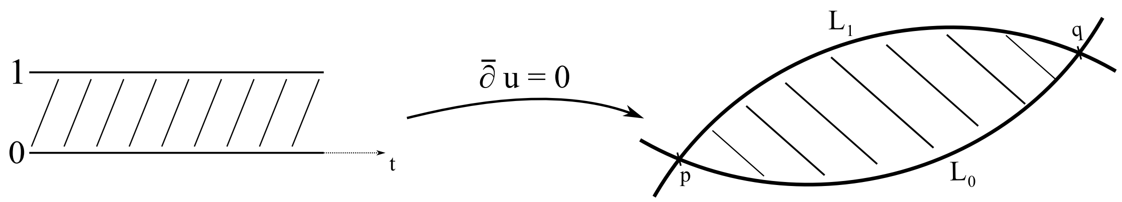


Figure 1.1: Examination of Lagrangian submanifolds by using pseudo-holomorphic curves

In [FloeI] A. Floer gave a proof that the following results

(i) $\partial \circ \partial = 0 \Rightarrow HF(L_0, L_1) := \ker \partial / \mathrm{im}\ \partial$ is defined

(ii) $L_0 \pitchfork L_1 \Rightarrow \#\{p \in L_0 \cap L_1\} \geq \sum_k HF^k(L_0, L_1)$

(iii) $HF(L_0, L_1) \cong HF(\phi_0(L_0), \phi_1(L_1))$ for ϕ_i being a Hamiltonian diffeomorphism

(iv) $HF(L, L) \cong H(L; \mathbb{Z}_2)$

can be achieved in the case

$$L_0 = L \ , \ L_1 = \phi(L) \tag{1.8}$$

for ϕ being a Hamiltonian diffeomorphism on M and L being Lagrangian submanifold in M satisfying

$$\pi_2(M, L) = 0 \ . \tag{1.9}$$

Combining the results (i)-(iv) provides an Arnold conjectural type result for Lagrangian intersections:

$$\#\{p \in L \cap \phi(L)\} \geq \sum_k \underbrace{b_k}_{\text{rank } H^k(L;\mathbb{Z}_2)} \tag{1.10}$$

In works of several others, one tried to achieve similar statements like (i)-(iv) for more general cases, not covered by 1.8 and 1.9. The most remarkable of these approaches was achieved by Y.-G. Oh in [Oh]. There he could prove the statements (i)-(iii) for monotone Lagrangian submanifolds, that is

$$c_1(\alpha) = \lambda \int_\alpha \omega \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad \alpha \in \operatorname{im}(\pi_2(M, L) \to H_2(M, L)), \tag{1.11}$$

with minimal Maslov number ≥ 3.

The sense, of how to get a more elaborate view on those described Lagrangian intersection issues, soon developed into evolving a more general description. The goal was to organize the present facts in a well understood algebraic language which can be handled properly and finally outputs a cohomology theory.
The major breakthrough in this direction was accomplished by K. Fukaya et al. in [FOOO1]. More precisely, by adopting the notion of A_∞-algebras, considered for the first time by J. Stasheff in [Sta], and extending it the filtered ones, they derive in Theorem A of [FOOO1]:

Theorem 1.1

The setup of a relatively spin (see (5.83)) Lagrangian submanifold $L \subset M$ can be formalized to a description in terms of a filtered A_∞-algebra.

By distinguishing between one 'output' and k 'input' points for pseudo-holomorphic discs with $k + 1$ marked points, they defined homomorphisms that map k given chains of L onto one, given by a perturbed moduli space $\mathcal{M}^{\mathfrak{s}}$. By dualizing the whole picture one gets

$$\underbrace{H^*(L; \Lambda) \otimes ... \otimes H^*(L; \Lambda)}_{k} \xrightarrow{m_k} H^*(L; \Lambda) \tag{1.12}$$

for appropriate ground rings Λ (see section 2.1 for a discussion of these so called Novikov rings). By now examining the boundary components of $\mathcal{M}$ (see chapter 5 for details), they could proof that these mentioned homomorphisms $\{m_k\}_{k\geq 0}$ fulfill they necessary A_∞-relation (3.63).
Generally speaking their approach is that powerful since they achieved to work with a permanent interplay between geometry and algebra. By organizing the $L^n \subset M^{2n}$ setup, via making use of the moduli space of pseudo-holomorphic curves, in an A_∞ manner, one can face present geometric complicacies by means of well understood algebraic concepts. After doing the necessary job on the algebra side we can go back to geometry (as we discuss in section 3.2.2 mostly by using the potential function

$\mathfrak{PD}$), equipped with a better insight into the behavior of Lagrangian submanifolds. Independently of the integration of A_∞-algebras into Floer theory, also physicists (mainly inspired by [Wi]) realized that these concepts can be used for the exploration of open string theory. In this thesis we are not discussing this relation to physics but refer to future research projects for details.

So what are A_∞-algebras and especially why do they incorporate such a generality? The rigorous description is postponed to chapter 3, here in this "down-to-earth definition" we just aim to provide a first feeling for how they can be used.

For a $\mathbb{Z}$-graded vector space $A = \bigoplus_{m\in\mathbb{Z}} A^m$ over a field R and degree $+1$ homomorphisms

$$m = \{m_k : \underbrace{A \otimes ... \otimes A}_{k} \to A\}_{k\geq 1 \text{ or } k\geq 0} \tag{1.13}$$

we say that (A, m) carries the structure of a (filtered) A_∞-algebra if for all k the A_∞-relation

$$\begin{aligned} \sum_{k_1+k_2=k+1} \sum_l (-1)^{\deg x_1+...+\deg x_{l-1}+l-1} \\ m_{k_1}(x_1, ..., x_{l-1}, m_{k_2}(x_l, ..., x_{l+k_2-1}), x_{l+k_2}, ..., x_k) = 0 \end{aligned} \tag{1.14}$$

is satisfied. The fact whether we consider $k \geq 1$ or $k \geq 0$ is quite crucial in the ongoing of this text since it reflects the cruciality if we can easily ($k \geq 1$) find a coboundary operator or if we have to work hard ($k \geq 0$) to do so.
To provide an intuition for the power of the A_∞-relation we first assume $k \geq 1$ and examine equation (1.14) for the cases $k = 2, 3$.

For $k = 2$ (1.14) writes as

$$m_1(m_2(x, y)) + m_2(m_1(x), y) + (-1)^{\deg x+1} m_2(x, m_1(y)) = 0 \tag{1.15}$$

and thus by defining

$$\begin{aligned} m_1(x) &=: dx \\ m_2(x, y) &=: (-1)^{\deg x-1} x \cdot y \end{aligned} \tag{1.16}$$

this equation can be rewritten as

$$\begin{aligned} 0 =& d(m_2(x, y)) + (-1)^{\deg m_1(x)-1} m_1(x) \cdot y + (-1)^{\deg x+1+\deg x-1} x \cdot m_1(y) \\ =& (-1)^{\deg x-1} d(x \cdot y) + (-1)^{\deg x+1-1} dx \cdot y + (-1)^{2\deg x} x \cdot dy \ . \end{aligned} \tag{1.17}$$

By multiplying with $(-1)^{\deg x}$ we conclude

$$d(x \cdot y) = dx \cdot y + (-1)^{\deg x} x \cdot dy \tag{1.18}$$

that is d, arising from m_1, is a differential that respects the graded Leibniz rule with respect to the multiplication $\cdot$.

For $k = 3$ (1.14) further yields:

$$\begin{aligned} 0 &= m_2(m_2(x,y),z) + (-1)^{\deg x+1} m_2(x, m_2(y,z)) + \text{"terms involving } m_3\text{"} = \\ &= (-1)^{\deg m_2(x,y)-1} m_2(x,y) \cdot z + (-1)^{2 \deg x} x \cdot m_2(y,z) + \cdots = \\ &= (-1)^{2 \deg x + \deg y + 1 - 1 - 1} (x \cdot y) \cdot z + (-1)^{2 \deg x + \deg y - 1} x \cdot (y \cdot z) + \cdots \end{aligned} \tag{1.19}$$

and thus

$$(x \cdot y) \cdot z + x \cdot (y \cdot z) = (-1)^{\deg y - 1} \text{ "terms involving } m_3\text{" } . \tag{1.20}$$

This last equation is read that the m_3 term measures the deviation of the multiplication $\cdot$, arising from m_2, from being associative.
In summary we have that A_∞-algebras can be seen as generalizations of differential graded algebras (D.G.A.). Precisely speaking we have the following inclusions:

$$\begin{aligned} &\{\text{D.G.A.}\} \subset \{A_\infty\text{-algebra}\} \\ &(A = \bigoplus_{m \in \mathbb{Z}} A^m, \cdot, d) \hookrightarrow (A, \{m_1(x) := dx, m_2(x,y) := (-1)^{\deg x - 1} x \cdot y, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m_0 = m_{i \geq 3} := 0\}) \\ \\ &\{\text{cochain complex}\} \subset \{A_\infty\text{-algebra}\} \\ &(A = \bigoplus_{m \in \mathbb{Z}} A^m, d) \hookrightarrow (A, \{m_1(x) := dx, m_0 = m_{i \geq 2} := 0\}) \end{aligned} \tag{1.21}$$

Not only considerations about generalizing the concept of differential graded algebras, but further information concerning Lagrangian Floer theoretic issues can be described A_∞-algebras.

As already announced above, we remark that the devil is in the detail, namely in the fact if one considers the $k \geq 1$ (unfiltered, see section 3.1.1) or the $k \geq 0$ (filtered, see section 3.1.1) case. The additional considerations about extending the theory to "filtered" A_∞-algebras become necessary since for the build up of Lagrangian Floer theory we have to consider the latter case. This fact reflects the appearance of holomorphic disc bubbles with no input and one output point and leads to the anomaly occurrence

$$\delta \circ \delta \neq 0 \; . \tag{1.22}$$

Speaking in A_∞ terms (with $k \geq 0$) this phenomenon is described by considering (1.14) for $k = 1$:

$$m_1(m_1(x)) + m_2(m_0(1), x) + (-1)^{\deg x + 1} m_2(x, m_0(1)) = 0 \tag{1.23}$$

One wants to declare the m_1 map as a coboundary operator, an assignment that is not possible yet due to (1.23) that in general yields

$$m_1 \circ m_1 \neq 0 \; . \tag{1.24}$$

In order to get rid of these anomalies we present two possible approaches of how to bypass them.
The "strict" solution (see section 3.2.1) would be to deform the maps $m_k \to m_k^b$ somehow in a way to achieve that $m_0^b \equiv 0$ and thus making the last two summands of (1.23) vanish.
Alternatively since the stated approach is in practice often not performable, the "weak" (see section 3.2.2) way out would be to achieve that the last last two summands of (1.23) do not vanish, but cancel each other and thus the desired result is achieved.
For both approaches one makes use of so called *(weak) Maurer-Cartan solutions b*. These are closely related to defining and examining a potential function $\mathfrak{PO}$, which is defined on the set formed by these elements. This arising function $\mathfrak{PO}$ is then used as a working tool for actually computing Lagrangian Floer Cohomology.

For not just mentioning buzzwords, we give a perspective on what is done in the present text.

Since notions appear throughout the whole text, chapter 1 recaps and summarizes basic facts about Novikov rings, later used as ground rings of our filtered A_∞-algebra, and the Maslov index for symplectic bundle pairs, which is used to better describe the appearing pseudo-holomorphic curves. These in turn form a moduli space $\mathcal{M}$, whose nature is also recapped in this convention chapter.

To lay the precise algebraic ground, chapter 3 describes the theory of (filtered) A_∞-algebras. For not drifting to deep into pure algebra, our focus hereby lies on concepts actually appearing in the algebraic description of geometry. We further discuss how the features of A_∞-algebras (mainly provided by the richness of information encoded in the A_∞-relation) can be used to define a coboundary operator and thus a cohomology theory (in this text denoted as Lagrangian Floer Cohomology). In this context it is quite natural to introduce the afore mentioned potential function $\mathfrak{PO}$ that will later on serve as a helpful tool when preforming computations.

Chapter 4, about the theory of Kuranishi structures, is quite technical and presents concepts that become important when formalizing geometric setups in the notion of A_∞-algebras. The process of transporting forms from a source space (later k copies of the Lagrangian submanifold L) via a space equipped with a Kuranishi structure (the perturbed moduli space $\mathcal{M}$ of pseudo-holomorphic curves attaching L) to a target space (the Lagrangian submanifold L) can be seen as the geometric appearance of the A_∞ homomorphisms $\{m_k\}_{k\geq 0}$.

After presenting notions in a general fashion so far, in chapter 5 we focus on toric symplectic manifolds. Essentially we fix notions and shortly recall how corresponding moment polytopes of certain selected examples (appearing in later considerations) arise. By incorporating the ideas of chapter 4, we show that the setup of Lagrangian torus fibers (over interior points of the moment polytope) in toric manifolds can be described in an A_∞-algebra fashion. One main advantage, of specifying to toric setup, is that certain A_∞ homomorphisms can actually be computed in terms of coordinate data of the underlying moment polytope of M. This works at least for those homomorphisms that appear in the definition of the potential function. It can be seen as major goal of this chapter to explicitly derive a coordinate description (provided by $\Delta \subset \mathbb{R}^n$) of $\mathfrak{PO}$ for Fano toric manifolds M.

We actually aim to face intersection issues of two Lagrangian submanifolds L_0, L_1. Such a setup a algebraically well described by light/left A_∞-bimodules over the A_∞-algebras associated to L_1 respectively L_0. Chapter 7 aims to provide a feeling for these concepts. On purpose we remain more sketchy here compared to how we presented the theory of A_∞-algebras in chapter 3. The reason for this kind of approaching is twofold. First it is mostly an algebraic generalization of ideas (by using standard Floer theoretic concepts) arising for A_∞-algebras and the focus of this text lies more in the treatment of geometry. Secondly, and even more important, Lagrangian Floer Cohomology, arising out of A_∞-bimodules for two Lagrangian submanifolds L_0, L_1, coincides with the one, defined for one Lagrangian submanifold L arising out of its corresponding A_∞-algebra, in the case $L_0 = L_1 = L$. With this background in mind we can ask for a lower bound on the number of intersection points of L with $\psi(L)$, that is the image of L when applying a Hamiltonian diffeomorphism ψ. A positive reply (depending on the Hofer norm $\|\psi\|$) into that direction is provided by a Theorem of K. Fukaya et al. (Theorem J in [FOOO1]) which only requires the knowledge of the Lagrangian Floer Cohomology of L. In the case of M being Fano toric we present a method how this in turn can be computed since, as we show there, the coboundary operator is computable in terms of derivatives of $\mathfrak{PO}$ (which we also know due to the results of chapter 5). We are closing with illustrating the introduced concepts by means of some examples.

Physicists call the appearing anomaly, m_1 not squaring up to 0, a BRST (BRST cohomology as the Hilbert space of physical states) symmetry breaking by soltion (disc bubbles) effects. Mainly the present thesis tries to explore the mathematical perspective, we though want to depict how notions naturally arise in topological string field theory. The usefulness of describing the theory in an A_∞ fashion shows up since we are again able to define a potential function, the *superpotential* Ψ, whose critical points form the moduli space of string configurations $\mathcal{M}$. In chapter 7 we explicitly show how cubic open string field theory can be described in terms of $A_\infty-$algebras. The thereof arising minimal model $(K, \{m_n\})$ is defined for the space of physical states K and the homomorphisms

$$m_n : K^{\otimes n} \to K \tag{1.25}$$

describe string products in tree-level Feynman diagrams. The motivation of this chapter can be seen as to (re-)state formerly appeared concepts, now regarding things from the perspective of a physicist.

Before finally starting with the main section of the present text we remark that the presented concepts mainly base upon the remarkable work of K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono. Out of their numerous works, dealing with Lagrangian

Floer theory and its influence on many mathematical branches like homological mirror symmetry etc., we are especially focusing on and making use of the ideas presented in [FOOO1], [FOOO2], [FOOO3], [FOOO4] and [FOOO5]. The intention for writing this thesis can be seen as providing a first feeling for the authors´ work by exemplifying some of their ideas in a detailed manner.

Chapter 2

Conventions and Working tools

Before diving into the conceptual buildup of an algebraic approach to Lagrangian Floer Cohomology, we first have to clarify some basic concepts and fix notations that will come up throughout the text.
Thereby we orient ourselves on the work of Fukaya et al. presented in [FOOO1].
For concepts about how to handle the underlying symplectic geometry we advert to the standard textbook [McSa]. Results concerning the theory of stable maps and compactification issues we lean ourselves mainly on [McSaII] that provides a detailed description in general and again [FOOO1] in order to stay consistent with the authors´ ideas and notations.

2.1 Novikov rings

As remarked in the introduction chapter 1 if one wants to apply the A_∞ machinery, for getting a more profound way of handling geometry, we can not easily get rid of the appearing m_0 terms. In order to control the arising infinite sums we have to find ways of how to complete the arising modules. This is done by using appropriate filtrations. The therefore necessary extension of standard A_∞-algebra (over R) structures to filtered ones is done by using Novikov rings $\Lambda_{0,nov}(R)$.
Most readers may somehow be familiar with these rings since they are already used in standard Floer theory. For M being Calabi-Yau they arise as the necessary ground ring of singular homology for being isomorphic to Floer homology.
Here we shortly recap how they are defined and explain how they can naturally equipped with a filtration $\mathcal{F}$ that naturally extends to a filtration on free graded modules over $\Lambda_{0,nov}(R)$.

For a commutative ring R with unit 1 (mostly the integers $\mathbb{Z}$ or the field $\mathbb{Q}$) and formal generators T (deg $T = 0$) and e (deg $e = 2$) the *universal Novikov ring* is defined as

$$\Lambda_{0,nov}(R) := \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} | a_i \in R, n_i \in \mathbb{Z}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i \leq \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty\} \ . \quad (2.1)$$

For R being a field $\Lambda_{0,nov}(R)$ is an integral domain and we denote its field of fraction (that is the smallest field containing $\Lambda_{0,nov}(R)$)

$$\begin{aligned}\mathrm{Frac}(\Lambda_{0,nov}(R)) &= \{\frac{a}{b}|a,b\in\Lambda_{0,nov}(R);b\neq 0\} = \\ &= \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i}|a_i\in R, n_i\in\mathbb{Z},\lambda_i\in\mathbb{R},\lambda_i\leq\lambda_{i+1},\lim_{i\to\infty}\lambda_i=\infty\} = \\ &=: \Lambda_{nov}(R)\end{aligned} \tag{2.2}$$

as the *universal Novikov field*. In some cases one is forced to be a bit more general here and is required to take R just being a commutative ring (with unit 1). One needs to apply the localization procedure that is a generalization of the construction of the field of fraction. Remark that in these cases $\Lambda_{nov}(R)$ is not a field that is not all elements have inverses then.
The ideal

$$\Lambda^+_{0,nov}(R) \equiv \Lambda_{+,nov}(R) \tag{2.3}$$

of $\Lambda_{0,nov}(R)$ consisting of elements with λ_i strictly positve and the fact

$$\Lambda_{0,nov}(R)/\Lambda^+_{0,nov}(R) \cong R[e,e^{-1}] \tag{2.4}$$

will play an important role when we later perform so called *R-reductions*. As described in chapter 3, the stated construction allows to reduce filtered A_∞-algebras (over $\Lambda_{0,nov}(R)$) to 'classic' (that is unfiltered) ones over R. Since the parameter λ_i is later used to encode the energy (symplectic volume)

$$\int_\Sigma u^*\omega \tag{2.5}$$

of the pseudo-holomorphic curves u, (2.4) and so R-reduction symplectically speaking means that we mod out curves of positive energy.
Two additional conventions shall be fixed from now on:

- In some cases (especially the toric ones later on) we 'forget' the generator e indicated by neglecting the subscript "nov", that is

  $$\Lambda^{(+)}_{(0)}(R) := \{\sum_{i=0}^{\infty} a_i T^{\lambda_i}|\ \cdots\}\ . \tag{2.6}$$

  We further highlight that for R being a field we get that

  $$\Lambda_0(R) \equiv \Lambda_0^R \tag{2.7}$$

  is a principal ideal domain. This holds since all ideals of Λ_0^R are of the form (T^k) (for $k\geq 0$) and thus are principal. We profit of this when later making use of the universal coefficient theorem. For the cohomology of the n-dimensional torus,

  $$H^k(T^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\binom{n}{n-k}} & ,\ \text{for } 0\leq k\leq n \\ 0 & ,\ \text{else}\end{cases}\ , \tag{2.8}$$

with coefficients in the principal ideal domain Λ_0^R we then get

$$H^*(T^n;\Lambda_0^R) \cong \bigoplus_{k=0}^{n} (\Lambda_0^R)^{\binom{n}{n-k}} \cong (\Lambda_0^R)^{2n} \ . \tag{2.9}$$

- The specification, with which ring R we are working with is, mostly neglected in the case $R = \mathbb{C}$, that is we write

$$\Lambda^{\cdots}_{\cdots}(R = \mathbb{C}) \equiv \Lambda^{\cdots}_{\cdots} \ . \tag{2.10}$$

On the above described rings the map

$$\begin{aligned} \nu_T : \Lambda^{\cdots}_{\cdots}(R) &\to \mathbb{R} \\ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} &\mapsto \inf \{\lambda_i \mid a_i \neq 0\} \end{aligned} \tag{2.11}$$

is well-defined since we assume without loss of generality $(\lambda_i, n_i) \neq (\lambda_j, n_j)$ for $(i \neq j)$. It can be seen as a *non-Archimedean valuation*, that is it fulfills

(i) $\nu_T(a) = \infty \Leftrightarrow a = 0$

(ii) $\nu_T(ab) = \nu_T(a) + \nu_T(b)$

(iii) $\min\{\nu_T(a), \nu_T(b)\} \leq \nu_T(a+b) \leq \max\{\nu_T(a), \nu_T(b)\}$.

Using the preimages of ν_T we get a filtration $\mathcal{F}_T$ of subrings for $\Lambda_{0,nov}(R)$ meaning

$$\Lambda_{0,nov}(R) = F^0\Lambda_{0,nov}(R) \supset F^1\Lambda_{0,nov}(R) \supset F^2\Lambda_{0,nov}(R) \supset \cdots \tag{2.12}$$

defined as

$$F^\lambda\Lambda_{0,nov}(R) := \nu_T^{-1}([\lambda,\infty)) = T^\lambda \cdot \Lambda_{0,nov}(R) \quad \text{for} \quad \lambda \geq 0 \ . \tag{2.13}$$

Later in chapter 3 we use these ideas to define a filtration $\mathcal{F}$ on

$$C \otimes_R \Lambda_{0,nov}(R) \tag{2.14}$$

for C being a free graded R module. This is done by similarly using preimages of the valuation

$$\begin{aligned} \nu : C \otimes_R \Lambda_{0,nov}(R) &\to \mathbb{R} \\ \sum_i \underbrace{x_i}_{\in\ \Lambda_{0,nov}(R)} \overrightarrow{e_i} &\mapsto \inf \{\nu_T(x_i)\} \ . \end{aligned} \tag{2.15}$$

The completion with respect to this filtration (in section 3.1.2 we shortly recap how this is performed) is then denoted by

$$C \ \hat{\otimes}_R \ \Lambda_{0,nov}(R) \ . \tag{2.16}$$

2.2 Maslov index for symplectic bundle pairs

We aim to recap some facts (see e.g. [McSa] for details) about Lagrangian vector spaces respectively subbundles.
Recall that for a Lagrangian subspace L^n in the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i)$ one always finds a unique (up to $O(n)$ action) unitary Lagrangian frame $Z = \begin{pmatrix} X \\ Y \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^{2n}$ such that $\mathrm{im}(Z) = L$. Since

$$\begin{aligned} X^tY - Y^tX &= 0 \\ XX^t + YY^t &= 1 \end{aligned} \tag{2.17}$$

we have

$$\psi := \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in Sp(2n) \cap O(2n) \cong U(n) \tag{2.18}$$

and therefore $U := X + iY \in U(n)$. With the fact

$$L = \psi \cdot L_{\text{hor.}} \tag{2.19}$$

for $L_{\text{hor.}} := \{(x,y) \in \mathbb{R}^{2n} \mid y = 0\}$ we deduce that the we have an isomorphism

$$\{L \mid L \text{ Lagrangian subspace in } (\mathbb{R}^{2n}, \omega_0)\} =: \mathcal{L}(n) \cong U(n)/O(n) \tag{2.20}$$

The defined *Lagrangian Grassmannian* $\mathcal{L}(n)$ is a manifold of dimension

$$\dim U(n) - \dim O(n) = n^2 - \frac{n^2 - n}{2} = \frac{n(n+1)}{2} \tag{2.21}$$

that is clearly invariant under actions of $Sp(2n)$ $(Sp(2n).\mathcal{L}(n) = \mathcal{L}(n))$.
The standard Maslov index assigns to each loop

$$\gamma : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(n) \tag{2.22}$$

an integer in the following way:
For $\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ being an unitary Lagrangian frame for the Lagrangian subspace $\gamma(t)$ at fixed time t, the map

$$\begin{aligned} \widetilde{\rho} \circ \gamma : S^1 &\to S^1 \\ t &\mapsto \det \big(\underbrace{X(t) + iY(t)}_{\in U(n)}\big)^2 \end{aligned} \tag{2.23}$$

gets lifted to

$$\alpha : S^1 \to \mathbb{R} \tag{2.24}$$

defined by

$$\det\,(X(t)+iY(t))^2 = e^{2\pi i\alpha(t)}\ . \tag{2.25}$$

The Maslov index is now defined via

$$\mu(\gamma) := \alpha(1)-\alpha(0)\ \in \mathbb{Z} \tag{2.26}$$

that is the mapping degree of $\widetilde{\rho}\circ\gamma$.
The power factor 2 in (2.23) and thus in (2.25) comes into play since Lagrangians with reversed orientation are considered to be equal. This guarantees well-definedness of $\widetilde{\rho}\circ\gamma$ despite the $O(n)$ freedom that we have for $X(t), Y(t)$.
Amongst other nice properties, the Maslov index is identical for loops in $\mathcal{L}$ if and only if they are homotopic to each other. Another important property, especially with regard to the well-definedness of the Maslov index for symplectic bundle pairs, is

$$\mu(\Psi\gamma) = \mu(\gamma) + 2\mu(\Psi) \tag{2.27}$$

for $\Psi : \mathbb{R}/\mathbb{Z} \to Sp(2n)$ being a loop in the group of symplectic matrices. As remarked above we have $\Psi(t)\cdot\gamma(t) \in \mathcal{L}(n)$. Here $\mu(\Psi)$ is the mapping degree of

$$(\rho\circ\Psi)(t) := \det(\underbrace{X(t)+iY(t)}_{\in U(n)})\ . \tag{2.28}$$

For t fixed and thus a fixed symplectic matrix $\Psi(t) = \psi$, the right hand side is determined by $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = (\psi\psi^t)^{-1/2}\psi$ being the orthogonal part in the polar decomposition

$$\psi = P\cdot Q = (\psi\psi^t)^{1/2}\cdot \underbrace{(\psi\psi^t)^{-1/2}\psi}_{\in Sp(2n)\cap O(2n)}\ . \tag{2.29}$$

As in [FOOO1] we aim to assign a Maslov index to smooth maps

$$f : (\Sigma, \partial\Sigma) \to (M, L) \tag{2.30}$$

in order to get a better insight into the behavior of pseudo-holomorphic curves. Here Σ denotes a compact, oriented surface of genus g with $h \neq 0$ connected boundary components $\partial_i\Sigma$. L denotes a Lagrangian submanifold in (M^{2n}, ω). With

$$(f^*TM, f|_{\partial\Sigma}^*TL) =: (\chi, \lambda) \tag{2.31}$$

we define a *symplectic bundle pair* that is

$$\chi \to \Sigma,\ \ \chi|_{\partial\Sigma} \to \partial\Sigma \tag{2.32}$$

being rank $2n$ symplectic vector bundles and

$$\lambda \to \partial\Sigma \tag{2.33}$$

being a Lagrangian subbundle of the latter. By picking a compatible, almost complex structure on $\chi \to \Sigma$ it gets Hermitian and we can apply Proposition 2.66. of [McSa]. Since $\partial\Sigma$ is required to be non-empty, we find a unitary trivialization

$$\phi : \chi \to \Sigma \times (\mathbb{R}^{2n}, \omega_0)\ . \tag{2.34}$$

The restriction of the trivialization to one of the h boundary components $\lambda|_{\partial_i\Sigma}$ ($\partial_i\Sigma \cong S^1$) can thus be interpreted as a loop

$$\gamma^i_\phi : S^1 \to \mathcal{L}(n) \tag{2.35}$$

to which we assign our above in (2.26) defined Maslov index, which is denoted by $\mu(\phi, \partial_i\Sigma)$ in the following.
The Maslov index of f is defined as the sum, over the boundary components of Σ, of these, that is

$$\mu_L(f) := \sum_{i=1}^{h} \mu(\phi, \partial_i\Sigma) \ . \tag{2.36}$$

It remains to check that this definition is well defined, meaning that is independent of the chosen trivialization ϕ.
For given trivializations ϕ_1, ϕ_2 we have

$$\begin{aligned} \phi_2 \circ \phi_1^{-1} : \Sigma \times (\mathbb{R}^{2n}, \omega_0) &\to \Sigma \times (\mathbb{R}^{2n}, \omega_0) \\ (x, v) &\mapsto (x, \varphi(x)v) \end{aligned} \tag{2.37}$$

for $\varphi : \Sigma \to Sp(2n)$. For the above described loops we therefore get the relation

$$\varphi|_{\partial_i\Sigma}(x) \cdot \gamma^i_{\phi_1}(x) = \gamma^i_{\phi_2}(x) \ \in \mathcal{L}(n) \ . \tag{2.38}$$

With (2.27) we further deduce a relation between their Maslov indices, namely

$$\mu(\phi_2, \partial_i\Sigma) = \mu(\phi_1, \partial_i\Sigma) + 2\mu(\varphi|_{\partial_i\Sigma}) \tag{2.39}$$

and therefore

$$\sum_{i=1}^{h} \mu(\phi_2, \partial_i\Sigma) = \sum_{i=1}^{h} \mu(\phi_1, \partial_i\Sigma) + 2\underbrace{\sum_{i=1}^{h} \mu(\varphi|_{\partial_i\Sigma})}_{(*)} \stackrel{!}{=} \mu_L(f) \ . \tag{2.40}$$

So it remains to show that $(*)$ vanishes. This is true since the relevant angular part of $\varphi|_{\partial_i\Sigma}$ extends to the one of

$$\varphi : \Sigma \to Sp(2n) \ . \tag{2.41}$$

By then summing up the h relations

$$(\rho \circ \varphi)_*[\partial_i\Sigma] = \underbrace{\deg((\rho \circ \varphi)|_{\partial_i\Sigma})}_{\mu(\varphi|_{\partial_i\Sigma})} \cdot [S^1] \tag{2.42}$$

for the homomorphism $(\rho \circ \varphi)_* : H_1(\Sigma; \mathbb{Z}) \to H_1(S^1; \mathbb{Z})$ we get

$$(\rho \circ \varphi)_*(\underbrace{[\partial_1\Sigma] + ... + [\partial_k\Sigma]}_{[\partial\Sigma]}) = \sum_{i=1}^{h} \mu(\varphi|_{\partial_i\Sigma}) \cdot [S^1] \ . \tag{2.43}$$

Since $[\partial\Sigma] = 0 \in H_1(\Sigma; \mathbb{Z})$ we are done and the well-definedness of (2.36) is proven.

Remark 2.1. *For a given homotopy class*

$$\beta \in \pi_2(M, L) \tag{2.44}$$

we have a notion of symplectic volume and Maslov index, defined by

$$\omega(\beta) := \int_\Sigma u^*\omega \quad \textit{respectively} \quad \mu_L(\beta) := \mu_L(u), \tag{2.45}$$

for a pseudo-holomorphic curve

$$u : (\Sigma, \partial\Sigma) \to (M, L) \quad \textit{with} \quad [u] = \beta\ . \tag{2.46}$$

These are well-defined since the symplectic volume and the Maslov index of a pseudo-holomorphic curve u are homotopy invariants (see e.g. [McSaII]).

2.3 Moduli spaces of bordered stable maps

As already announced in the introductory chapter 1, a major tool for the study of Lagrangian submanifolds is the use of pseudo-holomorphic curves attaching them. We shortly recall notions appearing in that context.
A detailed discussion of these moduli space analytic issues can be found in [FOOO1]. As a general introduction to the subject of stable maps we recommend chapter 5 and 6 of [McSaII].

In the following we assume (M, ω) to be a symplectic manifold and $L \subset M$ being a Lagrangian submanifold.

Let $(\Sigma, \overrightarrow{z} = \{z_1, ..., z_l\}, \overrightarrow{z}^+ = \{z_1^+, ..., z_k^+\})$ be a *bordered marked stable Riemann surface of genus* $g = 0$, that is:

(i) Σ is a simply connected union of irreducible disc and sphere components Σ_i $(i \in I)$. The intersection of three components is empty, whereas two components intersect in either a point (denoted as *singular point*) or not at all. For two disc components their common singular point (if it exists) lies in the boundary of each. For a sphere and a disc component they intersect (if their intersection is not empty) in an interior point of the latter.

(ii) $z_1, ..., z_l \in \partial\Sigma$ are marked points, pairwise distinct and distinct from the singular points, called *boundary marked points.*

(iii) $z_1^+, ..., z_k^+ \in \mathring{\Sigma}$ are marked points, pairwise distinct and distinct from the singular points, called *interior marked points.*

For a given relative homology class

$$\beta \in H_2(M, L; \mathbb{Z}) \tag{2.47}$$

we consider pseudo-holomorphic curves

$$w : (\Sigma, \partial\Sigma) \to (M, L) \tag{2.48}$$

of class β meaning $w_*([\Sigma]) = \beta$.
For the maps w with restrictions $w|_{\Sigma_i} =: w_i$ we further require that one of the following conditions is fulfilled for each $i \in I$:

(i) w_i is not constant.

(ii) For Σ_i being a sphere component, the sum of the numbers of singular points and marked points of Σ_i is greater than three.
For Σ_i being a disc component, 2 times the sum of the numbers of interior singular points and interior marked points of Σ_i plus the sum of the numbers of boundary singular points and boundary marked points of Σ_i is greater than three.

Two such data $((\Sigma, \{z_1, ..., z_l\}, \{z_1^+, ..., z_k^+\}), w)$ and $((\Sigma', \{z_1', ..., z_l'\}, \{z_1'^+, ..., z_k'^+\}), w')$ are said to be isomorphic if

$$w' = w \circ \phi^{-1} \quad , \quad z_i' = \phi(z_i) \quad \text{and} \quad z_j'^+ = \phi(z_j^+) \tag{2.49}$$

for $\phi : \Sigma \to \Sigma'$ biholomorphic. These isomorphism classes form the moduli space of pseudo-holomorphic curves, denoted by $\mathcal{M}_{l,k}(\beta)$. It is possible to define a topology on $\mathcal{M}_{l,k}(\beta)$, that is used to compactify (details can be found in chapter 7 of [FOOO1]).
In the following we use

$$\mathcal{M}_{l,k}(\beta) \tag{2.50}$$

for the compact moduli space.
Further notations are commonly used in the following:

(i) $\mathcal{M}_{l,k=0}(\beta) \equiv \mathcal{M}_l(\beta)$

(ii) $\mathcal{M}_{l,k}^{\text{main}}(\beta)$ denotes the *main component*, that is the connected component (a choice of cyclic order of the boundary marked points determines a connected component) that contains

$$[((D^2, \{z_1, ..., z_l\}, \{z_1^+, ..., z_k^+\}), w)] \tag{2.51}$$

where the boundary marked points $z_i \in \partial D^2 = S^1$ are cyclically ordered with respect to the counter clockwise orientation of S^1.

(iii) $\mathcal{M}_l^{\text{main,reg}}(\beta) \subset \mathcal{M}_l^{\text{main}}(\beta)$ denotes the subset of maps from a disc.

(iv) Due to (2.49), the map

$$\begin{aligned} ev = (ev_1, ..., ev_l; ev_1^+, ..., ev_k^+) : \mathcal{M}_{l,k}^{\text{main}}(\beta) &\to L^l \times M^k \\ [((D^2, \{z_1, ..., z_l\}, \{z_1^+, ..., z_k^+\}), w)] &\mapsto (w(z_1), ..., w(z_l), w(z_1^+), ..., w(z_k^+)), \end{aligned} \tag{2.52}$$

called the *evaluation map*, is well defined.

Chapter 3

Algebraic Backbone of A_∞-structures

As stated in the introductory chapter one wants to organize the given setup of Lagrangian submanifolds in a general framework. According to the stated Theorem 1.1 one can revert to the rich framework of A_∞-structures. It captures the algebraic description of at least the case for L being relatively spin . The facts of the following chapter are relatively basic and find applications in different areas of mathematics (see e.g. 4.6) and physics (see e.g. 4.6). It does not mean that it is straightforward and to illustrating it in full generality would go beyond the scope of this thesis. So for not forgetting the main purpose of our work, namely to explore concrete geometric situations, we try to remain on a basic level. Basic in the sense of not plunging to deep into the strict algebra formalism. We rather pick out concepts that are actually needed for the ongoing and discuss more or less only notions that arise of geometric considerations. As far it is already possible at this stage of progress we try to motivate how ideas can be seen geometrically. Firstly we thereby aim to achieve a better readability for the geometrically orientated reader and secondly we can not even introduce some concepts without highlighting their geometric origin. Constructions like for example R-reduction can not be performed for general A_∞-algebras but are possible for those that arise out of the Lagrangian submanifold setup.
After a discussion of unfiltered (classical i.e. $m_0 = 0$) and then filtered A_∞-algebras we try to derive a cohomology theory out of them. If we would work with the classical ones this would be easy ($\delta := m_1$) but unfortunately since we want to describe geometry we have to ask for possible deformations of the given A_∞-algebra to achieve this goal. In section 3.2 we describe two different approaches to end up with an appropriate coboundary operator and then define a cohomology theory out of a given A_∞-algebra. In order to really achieve geometric results later on, we will work with a potential function $\mathfrak{PO}$ that can be defined in this context in an rigorous algebraic manner.
As in many parts of the text we are gearing to the work of K. Fukaya et al. especially [FOOO1].

3.1 (Un-/Filtered) A_∞-algebras

Since we treat both unfiltered and filtered A_∞-algebras in the following and in particular try to interrelate them somehow later on (see R-reduction in chapter 3.1.2), we highlight the former ones with a bar sign "$\overline{\cdot}$" as e.g. $(\overline{C}, \overline{m})$. For filtered A_∞-algebras we omit this mark and simply write (C, m).

3.1.1 Unfiltered A_∞-algebras

For a given free graded R module $\overline{C} = \bigoplus_{m\in\mathbb{Z}} \overline{C}^m$ (R unital commutative ring with unit 1) and its *shift* $\overline{C}[1]$ defined by $\overline{C}[1]^m := \overline{C}^{m+1}$ one associates

$$B_k(\overline{C}[1]) := \bigoplus_{m_1,...,m_k} (\overline{C}[1])^{m_1} \otimes ... \otimes (\overline{C}[1])^{m_k} \cong \overline{C}[1] \otimes ... \otimes \overline{C}[1]. \tag{3.1}$$

In the following, as part of the given data, it is endowed with homogeneous homomorphisms of degree $+1$

$$\begin{gathered} \overline{m}_k : B_k(\overline{C}[1]) \to \overline{C}[1] \ (k \geq 0) \\ \overline{m}_0 \equiv 0. \end{gathered} \tag{3.2}$$

The thereof defined *bar complex*

$$B(\overline{C}[1]) := \bigoplus_n B_n(\overline{C}[1]) \tag{3.3}$$

can be seen as a differential graded coalgebra with a degree 0 comultiplication

$$\Delta : B(\overline{C}[1]) \to B(\overline{C}[1]) \otimes B(\overline{C}[1]) \tag{3.4}$$

that is fulfilling $(\Delta \circ \mathrm{Id}) \circ \Delta = (\mathrm{Id} \circ \Delta) \circ \Delta$. Here Δ is given componentwise by

$$x_1 \otimes ... \otimes x_k \mapsto \sum_{i=1}^{n} (x_1 \otimes ... \otimes x_i) \otimes (x_{i+1} \otimes ... \otimes x_k). \tag{3.5}$$

Coderivations are defined by extending $\{\overline{m}_k\}_{k\geq 1}$ to degree $+1$ homomorphisms in the bar complex

$$\widehat{\overline{m}}_k : \bigoplus_n B_n(\overline{C}[1]) \to \bigoplus_n B_{n-k+1}(\overline{C}[1]) \tag{3.6}$$

which on the n-th component are defined via

$$\begin{aligned} x_1 \otimes ... \otimes x_n \mapsto \sum_{l=1}^{n-k+1} & (-1)^{\deg x_1 + ... + \deg x_{l-1} + l - 1} x_1 \otimes ... \\ & ... \otimes x_{l-1} \otimes \overline{m}_k(x_l, ..., x_{l+k-1}) \otimes x_{l+k} \otimes ... \otimes x_n \end{aligned} \tag{3.7}$$

One additionally requires

$$\widehat{\overline{m}}_k = 0 \quad \text{for} \quad k > n. \tag{3.8}$$

The *coderivation* (degree +1) $\widehat{\overline{d}} : B(\overline{C}[1]) \to B(\overline{C}[1])$ is then defined by

$$\widehat{\overline{d}} := \sum_{k=1}^{\infty} \widehat{\overline{m}}_k. \tag{3.9}$$

Definition 3.1 (*A_∞-algebra*)

The maps $\overline{m} = \{\overline{m}_k\}_{k\in\mathbb{Z}}$ define an (unflitered) A_∞-algebra $(\overline{C}, \overline{m})$ over R if the following identity

$$\widehat{\overline{d}} \circ \widehat{\overline{d}} = 0 \tag{3.10}$$

holds.

The requirement (3.10) can be rewritten in a form that is perhaps more convenient to readers who already got in contact with A_∞-algebras. It relates the formal access to A_∞-algebras by using the bar complex with the 'down-to-earth' motivation (1.1) presented in the introductory chapter. Since $\widehat{\overline{d}}$ serves a boundary map one has

$$\begin{aligned}
0 =& (\widehat{\overline{d}} \circ \widehat{\overline{d}})(x_1 \otimes ... \otimes x_n) = \\
=& \widehat{\overline{d}}\Big(\sum_{k_1=1}^{n} \widehat{\overline{m}}_{k_1}(x_1 \otimes ... \otimes x_n)\Big) = \\
=& \sum_{k_2=1}^{n-k_1+1} \widehat{\overline{m}}_{k_2}\Big(\sum_{k_1=1}^{n} \sum_{l_1=1}^{n-k_1+1} (-1)^{\deg x_1 + ... + \deg x_{l_1-1} + l_1 - 1} x_1 \otimes ... \\
& \qquad ... \otimes \overline{m}_{k_1}(x_{l_1}, ..., x_{l_1+k_1-1}) \otimes ... \otimes x_n\Big) = \\
=& \sum_{k_1+k_2\le n+1} \sum_{l_1=1}^{n-k_1+1} (-1)^{\deg x_1 + ... + \deg x_{l_1-1} + l_1 - 1} \widehat{\overline{m}}_{k_2}(x_1 \otimes ... \\
& \qquad \underbrace{... \otimes \overline{m}_{k_1}(x_{l_1}, ..., x_{l_1+k_1-1}) \otimes ... \otimes x_n}_{=: x'_1 \otimes ... \otimes x'_{n-k_1+1}}) = \\
=& \sum_{k_1+k_2\le n+1} \sum_{l_1=1}^{n-k_1+1} \sum_{l_2=1}^{n-k_1-k_2+2} (-1)^{\sum_{i=1}^{l_1-1}(\deg x_i + 1) + \sum_{j=1}^{l_2-1}(\deg x'_j + 1)} x'_1 \otimes ... \\
& \qquad ... \otimes \overline{m}_{k_2}(x'_{l_2}, ..., x'_{l_2+k_2-1}) \otimes ... \otimes x'_{n-k_1+1}.
\end{aligned} \tag{3.11}$$

By symmetry reasons we see that only the $l_2 = 1$ term survives. That is by denoting $l_1 \equiv l$ the last line and thus $0 = \widehat{\overline{d}} \circ \widehat{\overline{d}}$ can be rewritten as the *unfiltered A_∞-relation*:

$$\boxed{\begin{aligned} &\sum_{k_1+k_2=n+1} \sum_{l=1}^{n-k_1+1} (-1)^{\deg x_1 + ... + \deg x_{l-1} + l - 1} \\ &\qquad \overline{m}_{k_2}(x_1, ..., \overline{m}_{k_1}(x_l, ..., x_{l+k_1-1}), ..., x_n) = 0 \quad \text{for all} \quad n \ge 1 \end{aligned}} \tag{3.12}$$

Remark that $\overline{m}_{k=0} = 0$ holds in the unfiltered A_∞-algebra case. One thus gets amongst others

$$\begin{gathered}\overline{m}_1 \circ \overline{m}_1(x) = 0 \\ \text{for all } x \in B_1(\overline{C}[1]) = \overline{C}[1]\end{gathered} \tag{3.13}$$

for the case $n = 1$. So we have found a coboundary operator that gives a cochain complex $(\overline{C}[1], \overline{m}_1)$.

To be able to change from one A_∞-algebra to another, we consider maps

$$\overline{f} = \{\overline{f}_k\}_{k>0} : (\overline{C}_1, \overline{m}^1) \to (\overline{C}_2, \overline{m}^2) \tag{3.14}$$

of degree 0 acting as:

$$\begin{gathered}\overline{f}_k : B_k(\overline{C}_1[1]) \to \overline{C}_2[1] \\ \text{with } \overline{f}_0 \equiv 0\end{gathered} \tag{3.15}$$

A given set of such maps can be canonically extended to a homomorphism acting on the direct sum $B(\overline{C}[1]) := \bigoplus_n B_k(\overline{C}[1])$ by setting:

$$\begin{aligned}\widehat{\overline{f}} : B(\overline{C}_1[1]) &\to B(\overline{C}_2[1]) \\ \underbrace{x_1 \otimes ... \otimes x_k}_{\in B_k(\overline{C}[1])} \mapsto \sum_{0<k_1<...<k_n<k} &\overline{f}_{k_1}(x_1,..., x_{k_1}) \otimes ... \otimes \overline{f}_{k_{i+1}-k_i}(x_{k_i+1}, ..., x_{k_{i+1}}) \otimes ... \\ &... \otimes \overline{f}_{k-k_n}(x_{k_n+1}, ..., x_k)\end{aligned} \tag{3.16}$$

According to how $\widehat{\overline{f}}$ is defined here we can consider it as a coalgebra homomorphism between the differential graded coalgebras

$$(B(\overline{C}_1[1]), \Delta_1, \widehat{\overline{d}}_1) \to (B(\overline{C}_2[1]), \Delta_2, \widehat{\overline{d}}_2) \tag{3.17}$$

that is it fulfills $\Delta_2 \circ \widehat{\overline{f}} = (\widehat{\overline{f}} \otimes \widehat{\overline{f}}) \circ \Delta_1$.

Definition 3.2 (*A_∞-homomorphism*)

A family of degree 0 maps

$$\overline{f} = \{\overline{f}_k\}_{k>0} : (\overline{C}_1, \overline{m}^1) \to (\overline{C}_2, \overline{m}^2) \tag{3.18}$$

between A_∞-algebras $(\overline{C}_i, \overline{m}^i)$ $(i = 1, 2)$ is called an A_∞-homomorphism if the identity

$$\widehat{\overline{f}} \circ \widehat{\overline{d}}^1 = \widehat{\overline{d}}^2 \circ \widehat{\overline{f}} \tag{3.19}$$

is satisfied.
For given A_∞-homomorphisms $\overline{f}^i : (\overline{C}_i, \overline{m}^i) \to (\overline{C}_{i+1}, \overline{m}^{i+1})$ $(i = 1, 2)$ their

composition $\{(\overline{f}^2 \circ \overline{f}^1)_k\}_{k\geq 1}$ *is defined via*

$$(\overline{f}^2 \circ \overline{f}^1)_k(x_1,...,x_k) = \sum_m \sum_{k_1+...+k_m=m} \overline{f}^2_m(\overline{f}^1_{k_1}(x_1,...,x_{k_1}),...,\overline{f}^1_{k_m}(x_{k-k_m+1},...,x_k)). \tag{3.20}$$

Lemma 3.1

The set of A_∞-homomorphisms is closed under composition (if it is defined!). This means for A_∞-homomorphisms $\overline{f}^i : (\overline{C}_i, \overline{m}^i) \to (\overline{C}_{i+1}, \overline{m}^{i+1})$ $(i = 1,2)$ the composition

$$\overline{f}^2 \circ \overline{f}^1 : (\overline{C}_1, \overline{m}^1) \to (\overline{C}_3, \overline{m}^3) \tag{3.21}$$

is again a A_∞-homomorphism.

Proof: The important aspect of the proof is that the extension $(\widehat{\overline{f}^2 \circ \overline{f}^1})$ of the composition coincides with the composition $\widehat{\overline{f}}^2 \circ \widehat{\overline{f}}^1$ of the extensions. This fact is provided by

$$\begin{aligned}
&(\widehat{\overline{f}^2 \circ \overline{f}^1})(x_1 \otimes ... \otimes x_k) = \\
&= \sum_{0<k_1<...<k_n<k} (\overline{f}^2 \circ \overline{f}^1)_{k_1}(x_1,...,x_{k_1}) \otimes ... \otimes (\overline{f}^2 \circ \overline{f}^1)_{k_{i+1}-k_i}(x_{k_i+1},...,x_{k_{i+1}}) \otimes ... \\
&\qquad ... \otimes (\overline{f}^2 \circ \overline{f}^1)_{k-k_n}(x_{k_n+1},...,x_k) = \\
&= \sum_{\substack{0<k_1<...\\<k_n<k}} \left(\sum_{l_1} \sum_{k_{1_1}+...+k_{1_{l_1}}=l_1} \overline{f}^2_{l_1}(\overline{f}^1_{k_{1_1}}(...),...,\overline{f}^1_{k_{1_{l_1}}}(...)) \right) \otimes ... \\
&\qquad ... \otimes \left(\sum_{l_{(k-k_n)}} \sum_{\substack{(k-k_n)_1+...\\...+(k-k_n)_{l_{(k-k_n)}}=l_{(k-k_n)}}} \overline{f}^2_{l_{k-k_n}}(\overline{f}^1_{(k-k_n)_1}(...),...,\overline{f}^1_{(k-k_n)_{l_{k-k_n}}}(...)) \right) = \\
&= ... = \\
&= \sum_{\substack{0<l_1<...\\<l_i<k-k_n}} \sum_{\substack{0<k_1<...\\<k_n<k}} \overline{f}^2_{l_1}(\overline{f}^1_{k_1}(...),...,\overline{f}^1_{l_1}(...)) \otimes ... \otimes \overline{f}^2_{(k-k_n)-l_i}(\overline{f}^1_{l_i+1}(...),...,\overline{f}^1_{k-k_n}(...)) = \\
&= \widehat{\overline{f}}^2 \circ \widehat{\overline{f}}^1(x_1,...,x_k)
\end{aligned} \tag{3.22}$$

Now the claim

$$\begin{aligned}
(\widehat{\overline{f}^2 \circ \overline{f}^1}) \circ \widehat{\overline{d}}^1 &= \\
&= \widehat{\overline{f}}^2 \circ \widehat{\overline{f}}^1 \circ \widehat{\overline{d}}^1 = \widehat{\overline{f}}^2 \circ \widehat{\overline{d}}^2 \circ \widehat{\overline{f}}^1 = \widehat{\overline{d}}^3 \circ \widehat{\overline{f}}^2 \circ \widehat{\overline{f}}^1 = \\
&= \widehat{\overline{d}}^3 \circ (\widehat{\overline{f}^2 \circ \overline{f}^1})
\end{aligned} \tag{3.23}$$

holds trivially. ■

Remark 3.1. *:*

(i) As already remarked above equation (3.10) implies that $\widehat{\overline{d}}^i$ *can be seen as a coboundary operator for the cochain complexes* $(B(\overline{C}_i[1]), \widehat{\overline{d}}^i)$ $(i = 1, 2)$. *According to (3.19) the extended map* $\widehat{\overline{f}}$ *is a cochain map between these complexes. This shows that*

$$\overline{f} : (\overline{C}_1, \overline{m}^1) \to (\overline{C}_2, \overline{m}^2) \tag{3.24}$$

induces a homomorphism on the cohomological level mapping

$$\underbrace{H^*(B(\overline{C}_1[1]); R)}_{:=ker\, \widehat{\overline{d}}^1 / im\, \widehat{\overline{d}}^1} \xrightarrow{\widehat{\overline{f}}^*} H^*(B(\overline{C}_2[1]); R). \tag{3.25}$$

(ii) According to the required definition (3.19) we know that the images of

$$x_1 \otimes ... \otimes x_k \in B_k(\overline{C}_1[1]) \tag{3.26}$$

under $\widehat{\overline{f}} \circ \widehat{\overline{d}}^1$ *and* $\widehat{\overline{d}}^2 \circ \widehat{\overline{f}}$ *coincide that is in particular for* $k = 1$*:*

$$\widehat{\overline{f}}_1(\overline{m}_1^1(x)) = \overline{m}_1^2(\widehat{\overline{f}}_1(x)) \tag{3.27}$$

We thus can conclude that

$$\widehat{\overline{f}} \circ \widehat{\overline{d}}^1 = \widehat{\overline{d}}^2 \circ \widehat{\overline{f}} \Rightarrow \overline{f}_1 \circ \overline{m}_1^1 = \overline{m}_1^2 \circ \overline{f}_1 \tag{3.28}$$

According to (3.13) we have $\overline{f}_1$ *a cochain map between the cochain complexes* $(\overline{C}_1[1], \overline{m}_1^1)$ *and* $(\overline{C}_2[1], \overline{m}_1^2)$. *As in (i) we thus have a homomorphism*

$$\underbrace{H^*(\overline{C}_1[1]; R)}_{:=ker\, \overline{m}_1^1 / im\, \overline{m}_1^1} \xrightarrow{\overline{f}_1^*} H^*(\overline{C}_2[1]; R). \tag{3.29}$$

An A_∞*-homomorphism is further specified to be a* weak homotopy equivalence *(equivalently denoted as an* A_∞-deformation*) if the cochain map* $\overline{f}_1$ *between the two complexes* $(\overline{C}_1[1], \overline{m}_1^1)$, $(\overline{C}_2[1], \overline{m}_1^2)$ *gives rise to a cochain homotopy equivalence. That is it exists an appropriate* $\overline{f}_1^{-1}$ *and* $h_i : \overline{C}_i^k[1] \to \overline{C}_i^{k-1}[1]$ *such that:*

$$\begin{aligned} (\overline{f}_1^{-1} \circ \overline{f}_1) - Id_{\overline{C}_1[1]} &= h_1 \circ \overline{m}_1^1 + \overline{m}_1^1 \circ h_1 \\ (\overline{f}_1 \circ \overline{f}_1^{-1}) - Id_{\overline{C}_2[1]} &= h_2 \circ \overline{m}_1^2 + \overline{m}_1^2 \circ h_2 \end{aligned} \tag{3.30}$$

For R *being* $\mathbb{Z}$ *or a field this is equivalent to that (3.29) is an isomorphism.*

3.1.2 Filtered A_∞-algebras

In the following we want to be a bit more specific and choose $\Lambda_{0,nov}(R)$ (see definition (2.1)) as our ground ring of the free graded module $\bigoplus\limits_{m\in\mathbb{Z}} C^m$. The filtration

$$F^\lambda \Lambda_{0,nov}(R) := \nu_T^{-1}([\lambda,\infty)) = T^\lambda \cdot \Lambda_{0,nov}(R) \quad \text{for} \quad \lambda \geq 0 \tag{3.31}$$

on $\Lambda_{0,nov}(R)$ as discussed in section 2.1 can now be used to endow C^m with a filtration.

Definition 3.3

A filtration $\mathcal{F}$ of the form $F^\lambda C^m$ on the subgroups C^m of the free graded $\Lambda_{0,nov}$ module $\bigoplus\limits_{m\in\mathbb{Z}} C^m$ is called an energy filtration if the following is fulfilled:

(i) $F^\lambda C^m \subset F^{\lambda'} C^m$ *if* $\lambda > \lambda'$

(ii) $T^{\lambda_0} \cdot F^\lambda C^m \subset F^{\lambda+\lambda_0} C^m$

(iii) $e^k \cdot C^m \subset C^{m+2k}$

(iv) C^m *is complete with respect the* $\mathcal{F}$

(v) $\exists\, v_i \in F^0 C^m \setminus \bigcup\limits_{\lambda>0} F^\lambda C^m$ *s.th.* $\{v_i\}_{i\in I}$ *form a basis of* C^m

The completion of $\bigoplus\limits_{m\in\mathbb{Z}} C^m$ with respect to $\mathcal{F}$ is denoted by C.

An explanation is in order. Recall that T and e are degree 0 respectively degree 2 generators, so the requirements how the grade of C^m changes in (ii) and (iii) make sense. The manner how a multiplication with T alters the index λ of the filtration is reasonable since as described in (2.15) such an energy filtration can be defined by

$$F^\lambda C^m := T^\lambda \cdot C^m \quad \text{for} \quad \lambda \geq 0. \tag{3.32}$$

This definition justifies why we call $\mathcal{F}$ an energy filtration. It is characterized by the superscript λ which later on is used to encode the symplectic energy of the pseudo-holomorphic curves into the concept of filtered A_∞-algebras namely

$$\lambda \equiv \omega(\beta) = \int\limits_\Sigma u^*\omega \quad \text{for} \quad u_*[\Sigma] = \beta. \tag{3.33}$$

The meaning of completeness with respect to a given filtration can either be described by using inverse limits or more intuitively by defining $U \subset C^m$ to be an neighborhood of 0 if and only if it exists λ' such that

$$U \supset F^{\lambda'} C^m. \tag{3.34}$$

A given sequence $(x_i)_{i\in I} \subset C^m$ is now specified to be a Cauchy sequence if for all neighborhoods U of 0 one finds a $s_U \in I$ such that

$$x_\mu - x_\nu \in U \tag{3.35}$$

for all $\mu, \nu \geq s_U$. The meaning of convergence and completeness thus follows analogously. As usually we complete a given space by considering its completion as the space consisting of equivalence classes of Cauchy sequences of the former. For a more detailed description the reader is referred to e.g. [AM].

Similarly to the unfiltered case one considers families of degree +1 homomorphisms

$$\{m_k : B_k(C[1]) \to C([1])\}_{k\geq 0} \tag{3.36}$$

with the same mapping behavior as in the unfiltered case. Remark that we include the $k = 0$ case now, more precisely the map

$$m_0 : \Lambda_{0,nov} \to C[1] \tag{3.37}$$

does not vanish and has to be considered. To guarantee convergence with respect to the energy filtration at a later stage of progress we require for $\{m_k\}_{k\geq 0}$ in addition

$$m_k(F^{\lambda_1}C^{m_1}, ..., F^{\lambda_k}C^{m_k}) \subseteq F^{\lambda_1+...+\lambda_k}C^{m_1+...+m_k-k+2} \tag{3.38}$$

and

$$m_0(1) \in F^{\lambda'}C[1] \quad \text{for} \quad \lambda' > 0. \tag{3.39}$$

Here with 1 we mean the unit of the corresponding ground ring $\Lambda_{0,nov}$. The filtration as above can be used to declare a filtration $F^\lambda B_k(C[1])$ on the associated bar complex $B_k(C[1])$ via:

$$F^\lambda B_k(C[1]) := \bigcup_{\lambda_1+...+\lambda_k \geq \lambda} \bigoplus_{m_1,...,m_k} F^{\lambda_1}C[1]^{m_1} \otimes ... \otimes F^{\lambda_k}C[1]^{m_k}. \tag{3.40}$$

This in turn allows to complete $B_k(C[1])$ to $\widehat{B}_k(C[1])$ and consider the **completed bar complex**

$$\widehat{B}(C[1]) := \{\sum_k \mathbf{x}_k \ | \ \mathbf{x}_k \in F^{\lambda_k}\widehat{B}_k(C[1]), \lim_{k\to\infty} \lambda_k = \infty\}. \tag{3.41}$$

Definition 3.4

The homomorphisms $\{m_k\}_{k\geq 0}$ give rise to define coderivations

$$\begin{aligned} \widehat{m}_k : \bigoplus_n B_n(C[1]) &\longrightarrow \bigoplus_n B_{n-k+1}(C[1]) \\ x_1 \otimes ... \otimes x_n &\mapsto \sum_{l=1}^{n-k+1} (-1)^{deg\ x_1+...+deg\ x_{l-1}+l-1} x_1 \otimes ... \otimes x_{l-1} \otimes \\ &\qquad \otimes m_k(x_l, ..., x_{l+k-1}) \otimes x_{l+k} \otimes ... \otimes x_n. \end{aligned} \tag{3.42}$$

For $k = 0$ we have

$$\begin{aligned} \widehat{m}_{k=0}(x_1 \otimes ... \otimes x_n) := \sum_{l=1}^{n+1} (-1)^{deg\ x_1+...+deg\ x_{l-1}+l-1} x_1 \otimes ... \otimes x_{l-1} \otimes \\ \otimes m_0(1) \otimes x_l \otimes ... \otimes x_n. \end{aligned} \tag{3.43}$$

Again we require

$$\widehat{m}_k = 0 \text{ for } k > n. \tag{3.44}$$

They give rise to an operator

$$\widehat{d} := \sum_{k=0}^{\infty} \widehat{m}_k : \widehat{B}(C[1]) \to \widehat{B}(C[1]) \tag{3.45}$$

Comparably to the unfiltered A_∞-algebra case we think of $\widehat{B}(C[1])$ as a "formal" differential graded coalgebra (formal to symbolize that we work with completions) with comultiplication

$$\Delta : \widehat{B}(C[1]) \to \widehat{B}(C[1]) \widehat{\otimes} \widehat{B}(C[1]) \tag{3.46}$$

analogously defined by

$$\Delta(x_1 \widehat{\otimes} ... \widehat{\otimes} x_n) := \sum_{i=1}^{n} (x_1 \widehat{\otimes} ... \widehat{\otimes} x_i) \widehat{\otimes} (x_{i+1} \widehat{\otimes} ... \widehat{\otimes} x_n). \tag{3.47}$$

The " $\widehat{\cdot}$ " over $\otimes$ symbolizes that

$$x_1 \widehat{\otimes} ... \widehat{\otimes} x_n \in \widehat{B}_n(C[1]). \tag{3.48}$$

To have $\widehat{d}$ as a comultiplication we are requiring $\widehat{d} \circ \widehat{d} = 0$ as a defining condition for filtered A_∞-algebras below.

Lemma 3.2

The image $im(\widehat{d})$ is contained in $\widehat{B}(C[1])$, that is $\widehat{d}$ is a well defined operator.

Proof: The l-th component of $\widehat{d}(\sum_k \mathbf{x}_k)$ in $\widehat{B}(C[1])$ is given by $\sum_i \widehat{m}_i(\mathbf{x}_{l+i-1})$. According to (3.38) and (3.39) one sees that $\widehat{m}_i(\mathbf{x}_{l+i-1}) \in F^{\lambda_{l+i-1}} \widehat{B}_l(C[1])$ and so with (3.41) one follows that $\widehat{d}(\sum_k \mathbf{x}_k)$ converges componentwise i.e. really lies in $\widehat{B}(C[1])$. ■

***R*-reduction:**

R-reduction provides a possibility to reduce the filtered $\Lambda_{0,nov}(R)$ module C to a unfiltered R module $\overline{C}$. Anticipating the later construction of a filtered A_∞-algebra for our geometric "$L \subset M$" setup we remark that we first construct an unfiltered A_∞-algebra over R (mostly $R = \mathbb{Q}$) and then extend it to a filtered one by tensoring it with $\Lambda_{0,nov}(R) \equiv \Lambda_{0,nov}$. We thus can always assume that both are linked by an appropriate isomorphism

$$C \cong \overline{C} \otimes_R \Lambda_{0,nov}. \tag{3.49}$$

The isomorphism

$$\Lambda_{0,nov}/\Lambda^+_{0,nov} \cong R[e, e^{-1}] (\text{see } 2.4) \tag{3.50}$$

is used to also reduce $\{m_k\}_{k\geq 0}$ to a family of homomorphisms $\{\overline{m}_k\}_{k\geq 1}$. We define

$$\overline{m}_k : B_k(\overline{C}[1]) \otimes_R R[e, e^{-1}] \to C[1] \otimes_R R[e, e^{-1}] \tag{3.51}$$

via

$$\overline{m}_k(x_1, ..., x_k) := m_k(x_1, ..., x_k) \mod \Lambda^+_{0,nov} C. \tag{3.52}$$

Assuming that the $\{m_k\}$ map to 0 if $R[e, e^{-1}]$ coefficients are involved we can deduce that they induce homomorphisms

$$\overline{m}_k : B_k(\overline{C}[1]) \to C[1] \tag{3.53}$$

(here we use the symbol $\overline{m}_k$ twice on purpose). The stated assumption clearly is wrong for general A_∞-algebras but again holds for the ones constructed out of geometry. Recall that the parameter λ is used to incorporate the energy of the pseudo-holomorphic curves. Thus (3.50) is read geometrically as modding out curves with positive energy that is we only regard constant ones. These in turn have Maslov index $\mu_L(\beta) = 0$ (see 2.36). The superscript parameter $n_i \in \mathbb{Z}$ of the generator e is used to adjoin the Maslov index that is we will put $\mu_L(\beta) \equiv n_i$ later. We conclude that $e^0 = 1$ and thus the assumption is justified for our cases later on. Needless to say all required relations transmit from m_k to $\overline{m}_k$. Remark that due to the requirement

$$m_0(1) \in F^{\lambda'} C[1] \quad \text{for} \quad \lambda' > 0. \tag{3.54}$$

(3.52) provides $\overline{m}_0 = 0$. In summary we get an unfiltered A_∞-algebra

$$(\overline{C}, \overline{m} = \{\overline{m}_k\}_{k\geq 1}) \tag{3.55}$$

over R called the *R-reduction* of $(C, m = \{m_k\}_{k\geq 0})$.

G-gappedness:

The outlined incorporation of $\omega(\beta)$ and $\mu_L(\beta)$ via the superscripts λ_i and n_i of the Novikov ring $\Lambda_{0,nov}$ allows to further specify A_∞-algebras by the property of being *G-gapped*. Here G denotes a submonoid

$$(G, +, (0, 0)) \subset (\mathbb{R}_{\geq 0} \times 2\mathbb{Z}, +, (0, 0)) \tag{3.56}$$

that is G contains the identity element $(0, 0)$ and is closed under the binary operation $+$. In addition the following requirements are posed on G:

Condition 3.1.

(i) Let be π the projection onto the first factor. Then $\pi(G) \subset \mathbb{R}_{\geq 0}$ lies discrete in $\mathbb{R}_{\geq 0}$.

(ii) $\forall p \in \mathbb{R}_{\geq 0}$: $G \cap (\{p\} \times 2\mathbb{Z})$ is finite.

(iii) $G \cap (\{0\} \times 2\mathbb{Z})$ consists just of the identity element $(0, 0)$.

General A_∞-algebras are said to be *G-gapped* if the m_k's decompose as

$$m_k = \sum_i T^{\lambda_i} e^{n_i} m_{k,i} \quad \text{for} \quad (\lambda_i, n_i) \in G \tag{3.57}$$

for $m_{k,i} : B_k(\overline{C}[1]) \to \overline{C}[1]$ being R module homomorphisms.
Due to how they arise, the A_∞-algebras that we geometrically construct later are G-gapped when setting $\lambda_i = \omega(\beta)$, $n_i = \mu_L(\beta)/2$, denoting $m_{k,i} \equiv m_{k,\beta}$ and summing over all possible $\beta \in H_2(M, L;)$ that can be realized by pseudo-holomorphic curves. Above we already discussed point (iii) namely that constant curves have vanishing Maslov index. Properties (i) (energies lie discrete) and (ii) (for fixed energies there is only a finite number of possible Maslov indices) can be deduced by Gromov´s compactness theorem. We further have $\mu_L(\beta) \in 2\mathbb{Z}$ since we consider oriented Lagrangians.

The preceding facts are summarized in the following definition:

Definition 3.5

(a) $(\bigoplus_{m\in\mathbb{Z}} C^m, m = \{m_k\}_{k\geq 0})$ *equipped with a filtration* $\mathcal{F}$ *of the form* $F^\lambda C^m$ *is said to be a filtered* A_∞*-algebra if* $\mathcal{F}$ *is an energy filtration and the following properties hold:*

  (i) *(3.38) and (3.39)*

  (ii) $\widehat{d} \circ \widehat{d} = 0$

(b) *The unfiltered* A_∞*-algebra* $(\overline{C}, \overline{m} = \{\overline{m}_k\}_{k\geq 1})$ *constructed out of* (C, m) *as in (3.49)-(3.53) is denoted as its R-reduction*

(c) *If*

$$m_0 \equiv 0 \tag{3.58}$$

(C, m) *is said to be strict.*

(d) (C, m) *is called unital with unit* $\mathbf{e}$ *if and only if*

$$\exists\ \mathbf{e} \in C[1]^{-1} = C^0 \tag{3.59}$$

satisfying:

  (i) $m_2(\mathbf{e}, x) = (-1)^{\deg(x)} m_2(x, \mathbf{e}) = x$

  (ii) $m_{k+1}(x_1, ..., \mathbf{e}, ..., x_k) = 0 \quad$ *for* $\quad k \in \{0, 2, 3, ...\}$

(e) *If the maps* $\{m_k\}_{k\geq 0}$ *decompose as*

$$m_k = \sum_i T^{\lambda_i} e^{n_i} m_{k,i} \tag{3.60}$$

for R module homomorphisms $m_{k,i} : B_k(\overline{C}[1]) \to \overline{C}[1]$ and

$$G = \{(\lambda_i, n_i)\} \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z} \tag{3.61}$$

satisfies condition 3.1, then (C, m) is called G-gapped.

(f) *Another A_∞-algebra (C', m') is an A_∞-deformation of (C, m) if there is an weak homotopy equivalence (see Remark 3.1 (ii))*

$$\overline{f} : (\overline{C}, \overline{m}) \to (\overline{C}', \overline{m}') \tag{3.62}$$

between their R-reductions.

Remark 3.2. *(i) For considering A_∞-deformations, remark that one first has to perform a R-reduction before being able to talk about weak homotopy equivalences. Recall that one needs m_1 to be a coboundary map for that notion. For filtered A_∞-algebras this does not hold in general due to the presence of the m_0 map but can be achieved by R-reducing them to unfiltered ones and thus gets $\overline{m}_1 \circ \overline{m}_1 = 0$.*

(ii) Similarly to (3.11) one can show that

$$\begin{aligned} &\widehat{d} \circ \widehat{d} = 0 \\ &\Leftrightarrow \\ &\sum_{k_1+k_2=n+1} \sum_l (-1)^{deg\ x_1 + \ldots + deg\ x_{l-1} + l - 1} \\ &\qquad m_{k_2}(x_1, \ldots, m_{k_1}(x_l, \ldots, x_{l+k_1-1}), \ldots, x_n) = 0 \quad \text{for all} \quad n \geq 0. \end{aligned} \tag{3.63}$$

The latter relation is known as the filtered A_∞-relation ($k_1, k_2 \geq 0$).

(iii) For $n = 1$ the above A_∞-relation writes as

$$m_1(m_1(x)) = -m_2(m_0(1), x) - (-1)^{deg\ x+1} m_2(x, m_0(1)). \tag{3.64}$$

In order to define Floer cohomology out of an A_∞-algebra and to use m_1 as a possible coboundary operator, one therefore has to find ways to make m_0 vanish. Such an approach is described in section (3.2.1) where we discuss ways how A_∞-algebras can be deformed into strict ones.
Since such deformations are mostly impossible to perform we follow another possibility in section (3.2.2) namely to define an appropriate coboundary operator in order to achieve that both terms on the right hand side cancel each other.

As in the unfiltered A_∞-algebra case one also considers the notion of filtered A_∞-homomorphisms between filtered A_∞-algebras. A full and proper treatment shall be omitted here. We only state the necessary facts that become important in later

sections. For a detailed description of that subject the interested reader is referred to the literature (e.g. [FOOO1]) where concepts are presented in full generality.
For given filtered A_∞-algebras (C_i, m^i) $(i = 1, 2)$ over $\Lambda_{0,nov}$ and a family of degree 0 homomorphisms

$$f = \{f_l : B_l(C_1[1]) \to C_2[1]\}_{l\geq 0} \tag{3.65}$$

we achieve convergence with respect to the filtration $\mathcal{F}$ when additionally requiring:

$$\begin{gathered} f_0(1) \in F^{\lambda'}C_2[1] \quad \text{for} \quad \lambda' > 0 \\ \text{and} \\ f_l(F^\lambda B_l(C_1[1])) \subseteq F^\lambda C_2[1] \end{gathered} \tag{3.66}$$

As in the last section we can thereby define a homomorphism $\widehat{f}$ (formal coalgebra homomorphism) between the associated completed bar complexes (formal differential graded coalgebras)

$$\widehat{B}(C_1[1]) \to \widehat{B}(C_2[1]). \tag{3.67}$$

On the l-th component of $\bigoplus_n B_n(C_1[1])$ it is of the form

$$\begin{aligned} \widehat{f}(x_1 \otimes ... \otimes x_l) = \sum_{0\leq l_1\leq ...\leq l_n\leq l} & f_{l_1}(x_1, ..., x_{l_1}) \otimes ... \otimes \\ & \otimes f_{l_{i+1}-l_i}(x_{l_{i+1}}, ..., x_{l_i+1}) \otimes ... \otimes f_{l-l_n}(x_{l_n+1}, ..., x_l). \end{aligned} \tag{3.68}$$

For the $l = 0$ case we set

$$\widehat{f}(1) = 1 + f_0(1) + f_0(1) \otimes f_0(1) + ... \,. \tag{3.69}$$

We remark that convergence (wrt. the energy filtration $\mathcal{F}$) of the right hand side of both equations is given here since we require (3.66) for $f = \{f_l\}_{l\geq 0}$.
The homomorphism

$$f = \{f_l\}_{l\geq 0} : (C^1, m^1) \to (C^2, m^2) \tag{3.70}$$

is said to be a *filtered A_∞-homomorphism* if

$$\boxed{\widehat{f} \circ \widehat{d}^1 = \widehat{d}^2 \circ \widehat{f}} \tag{3.71}$$

It is further specified to be a *strict filtered A_∞-homomorphism* if

$$f_0 \equiv 0. \tag{3.72}$$

The A_∞-homomorphism f is said to be *unital* if it "preserves" the unit. On the basis of Definition (3.5)(d) this means in this context

$$\begin{gathered} f_1(\mathbf{e}_1) = \mathbf{e}_2 \\ \text{and} \\ f_l(x_1, ..., x_{i-1}, \mathbf{e}_1, x_{i+1}, ..., x_l) = 0 \quad \text{for} \quad l \geq 2. \end{gathered} \tag{3.73}$$

For composing different A_∞-homomorphism one can make use of the analogue to Lemma (3.1). It naturally holds as well in the filtered case.

Lemma 3.3

The Composition of filtered A_∞-homomorphisms

$$f^i : (C_i, m^i) \to (C_{i+1}, m^{i+1}) \qquad i = 1, 2 \tag{3.74}$$

is defined by

$$(f^2 \circ f^1)_l(x_1, ..., x_l) = \sum_m \sum_{l_1+...+l_m=m} f^2_m(f^1_{l_1}(x_1, ..., x_{l_1}), ..., f^1_{l_m}(x_{l-l_m+1}, ..., x_l)). \tag{3.75}$$

The set of filtered A_∞-homomorphisms is closed under composition (if it is defined!). This means

$$f^2 \circ f^1 : (C_1, m^1) \to (C_3, m^3) \tag{3.76}$$

is again a filtered A_∞-homomorphism.

Proof: It works analogously to the proof of Theorem (3.1). The fact that we now work with $l \in \{0, 1, 2, ...\}$ instead of $k \in \{1, 2, ...\}$ affects on that the strict " $<$ " in the appearing sums get replaced by " $\leq$ " signs. ■

Based on the remarks about R-reduction (3.53) and G-gappedness of (3.57) A_∞-algebras we outline that these concepts can be examined for A_∞-homomorphisms.

Again by modding out the ideal $\Lambda^+_{0,nov}$, $\{f_k\}_{k\geq 0}$ boils down to an (unfiltered) A_∞-homomorphism

$$\{\overline{f}_k : B_k(\overline{C}_1[1]) \to (\overline{C}_2[1])\}_{k\geq 1} \tag{3.77}$$

over the ring R.

As already expected $\{f_k\}_{k\geq 0}$ is called *G-gapped* ($G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ submonoid) if it decomposes as

$$f_k = \sum_i T^{\lambda_i} e^{n_i} f_{k,i} \quad \text{for} \quad (\lambda_i, n_i) \in G \tag{3.78}$$

for $f_{k,i}$ being R module homomorphisms

$$B_k(\overline{C}_1[1]) \to (\overline{C}_2[1]). \tag{3.79}$$

Remark that both stated concepts appear when regarding filtered A_∞-algebras arising out of "$L \subset M$" geometry regards and setting $(\lambda_i, n_i) = (\omega(\beta), \mu_L(\beta)/2)$.

3.2 Search for A_∞-Maurer-Cartan solutions

As known in standard Floer theory, it is mostly the hard part to really find a possible boundary operator δ. Bubbling phenomena result in that the requirement

$$\delta \circ \delta = 0$$

does not hold in general. We are again faced with this kind problem when aiming to define *Floer Cohomology* out of given A_∞-algebras constructed in order to formalize the behavior of Lagrangian submanifolds. See section 5.3 for a discussion of possible ways for constructing the link between the geometry side we actually want to examine and the formal A_∞-algebra part. In this section we are still remaining to describe the algebraic point of view. Its motivation is to demonstrate how to use the features of A_∞-algebras in order to face and actually bypass the circumstances mentioned above. In the end appropriate deformations shall provide coboundary operations

$$\delta : C[1] \to C[1] \tag{3.80}$$

and therefore induce a cohomology theory.

As seen in (3.64) for degree $n = 1$ the A_∞-relation provides the identity

$$m_1(m_1(x)) + m_2(m_0(1), x) + (-1)^{\deg x+1} m_2(x, m_0(1)) = 0 \tag{3.81}$$

for

$$x \in B_{k=1}(C[1]) = C[1]. \tag{3.82}$$

When regarding equation (3.81) there are three more or less obvious ways how to reduce it to

$$m_1 \circ m_1 = 0 \tag{3.83}$$

that is to make the last two summands vanish and then declare

$$\delta := m_1 \tag{3.84}$$

as a coboundary operator for $C[1]$:

(i) Having $m_0(1) \equiv 0$ would be the most obvious way but is not possible in general since the maps $\{m_k\}$ are determined by the underlying geometry.

(ii) The idea that we follow in section 3.2.1 will be to use an already constructed A_∞-algebra (C, m) and try to deform it to another A_∞-algebra (C, m^b) for an appropriate $b \in C^1$. The meaning of b and m^b will be clarified in the stated section. The advantage of (C, m^b) now is that it is a strict (see Def. 3.5 (c)) one and we thus conclude that for (C, m^b) equation (3.81) is of the form

$$m_1^b \circ m_1^b = 0. \tag{3.85}$$

We then put $\delta := m_1^b$.
Such elements b will form the set of *strict Maurer-Cartan solution* $\widehat{\mathcal{M}}_{\text{strict}}(C)$ (see Def. 3.7).

(iii) Since it is often impossible to find such an element b necessary for (ii) we present a weaker version of (ii) in section (3.2.2). The idea will be to convert equation (3.81) in a way such that the last two summands cancel each other. To do so we try to find so called *weak Maurer-Cartan solution* $\widehat{\mathcal{M}}_{\text{weak}}(C)$ (see Def. 3.118).

The nice feature of the upcoming section 3.2 is that we can face geometric difficulties by purely algebraic methods already incorporated in the A_∞-algebras´ nature. Another tool that we thereof will gain is the *potential function* (see (3.126))

$$\mathfrak{PO} : \widehat{\mathcal{M}}_{\text{weak}}(C) \to \Lambda^+_{0,nov}. \tag{3.86}$$

We will see that

$$\widehat{\mathcal{M}}_{\text{strict}}(C) = \mathfrak{PO}^{-1}(0) \subset \widehat{\mathcal{M}}_{\text{weak}}(C). \tag{3.87}$$

As already noticed in the introductory chapter A_∞-algebras appear in the formulation of string field theory. Physicists also developed a notion of a potential function, the so called *superpotential*

$$\Psi : \Lambda_{0,nov} \times ... \times \Lambda_{0,nov} \to \Lambda_{0,nov}. \tag{3.88}$$

In addition it can surprisingly be used in mathematics in order to detect strict Maurer-Cartan solutions. In chapter 7 we try to highlight how $\mathfrak{PO}$ and Ψ are interrelated. Once more both disciplines thus can profit from one anothers´ work. As in most parts of the text we follow the ideas of K. Fukaya et al. presented in [FOOO1].

3.2.1 (Strict) Unobstructedness via Deformation

In the following we need the expression

$$e^b := 1 + b + b \otimes b + b \otimes b \otimes b + ... \quad . \tag{3.89}$$

In order to handle them properly we want these objects to live in the completed bar complex $\widehat{B}(C[1])$ (see (3.41)). To satisfy this requirement one needs

$$\underbrace{b \otimes ... \otimes b}_{l} \overset{!}{\in} F^{\lambda_l}\widehat{B}_l(C[1]) \quad \text{with} \quad \lim_{l\to\infty} \lambda_l = \infty. \tag{3.90}$$

According to the properties of the energy filtration (definition 3.3 and equations (3.38)-(3.39)) this is guaranteed amongst others for elements

$$b \in (C[1])^0 \quad \text{with} \quad b \equiv 0 \bmod \Lambda^+_{0,nov}. \tag{3.91}$$

That is one has $b \in F^\lambda C^1$ (for $\lambda > 0$) and thus with (3.40) we get

$$\underbrace{b \otimes ... \otimes b}_{l} \in F^{\lambda \cdot l}\widehat{B}_l(C[1]) \tag{3.92}$$

and therefore conclude

$$e^b \in \widehat{B}(C[1]). \tag{3.93}$$

As outlined above we want to deform the A_∞-homomorphisms m to m^b for such a b satisfying (3.91).
The upcoming definition describes how this modification is performed.

Definition 3.6

The deformation of an filtered A_∞-algebra (C, m) with b satisfying (3.91) is denoted by (C, m^b). The deformed homomorphisms

$$\{m_l^b : B_l(C[1]) \to C[1]\}_{l\geq 0} \tag{3.94}$$

are defined componentwise by

$$\begin{aligned} x_1, ..., x_l \mapsto \sum_{k_0,...,k_l} m_{l+\sum k_i}(\underbrace{b, ..., b}_{k_0}, x_1, \underbrace{b, ..., b}_{k_1}, ..., \underbrace{b, ..., b}_{k_{l-1}}, x_l, \underbrace{b, ..., b}_{k_l}) = \\ =: m(e^b x_1 e^b ... e^b x_l e^b). \end{aligned} \tag{3.95}$$

The next proposition enables us to apply the full A_∞-machinery and outline a sufficient condition posed on b for

$$m_1^b \circ m_1^b \tag{3.96}$$

to vanish identically.

Proposition 3.1

For (C, m) being a filtered A_∞-algebra, the graded module C with deformed maps $\{m_l^b\}_{l\geq 0}$ carries the structure of a filtered A_∞-algebra (C, m^b).
One has (C, m^b) to be a strict A_∞-algebra (i.e. $m_0^b = 0$) if and only if the corresponding $b \in (C[1])^0$, $b \equiv 0$ mod $\Lambda^+_{0,nov}$ fulfills

$$\widehat{d}(e^b) = 0. \tag{3.97}$$

Proof: For the first assertion we just have to check property (i), (ii) of definition 3.5. Since the module C is not affected by the deformation process, the condition on C^m to carry an energy filtration $F^\lambda C^m$ (see def. 3.3) carries over to the deformed case.
Due to the properties of $b \in F^\lambda C^1$ ($\lambda > 0$) and the primordial homomorphisms m_k, (3.39) holds for $m^b_{k=0}$ since one has ($\lambda, \lambda' > 0$)

$$m_0^b(1) = m(e^b) = \underbrace{m_0(1)}_{\in F^{\lambda'} C[1]^1} + \underbrace{m_1(b)}_{\in F^{\lambda\cdot 1} C^2} + \underbrace{m_2(b, b)}_{\in F^{\lambda\cdot 2} C^2} + ... \quad \in F^{\lambda''=\min\{\lambda,\lambda'\}>0} C[1]. \tag{3.98}$$

The argumentation works analogously for proving property (3.38), i.e.

$$m_l^b(\underbrace{x_1}_{\in F^{\lambda_1} C^{m_1}}, ..., \underbrace{x_l}_{\in F^{\lambda_l} C^{m_l}}) = \sum_{k_0,...,k_l} \underbrace{m_{l+\sum k_l}(b, ..., b, x_1, ..., x_l, b, ..., b)}_{\in F^{(k_0+...+k_l)\cdot\lambda + \lambda_1 + ... + \lambda_l} C^{k_1+...+k_l+m_1+...+m_l-(\sum k_i + l)+2}} \tag{3.99}$$

$$\subseteq F^{\lambda_1 + ... + \lambda_l} C^{m_1 + ... + m_l - l + 2}.$$

The best way to see that $\widehat{d^b}$ serves as a coboundary map for the completed complex $\widehat{B}(C[1])$ (i.e. to check $\widehat{d^b} \circ \widehat{d^b} = 0$) is to directly compare $\widehat{d^b}$ with $\widehat{d}$.
We have

$$
\begin{aligned}
0 &= (\widehat{d} \circ \widehat{d})(e^b x_1 e^b \otimes ... \otimes e^b x_n e^b) = \\
&= \widehat{d}\left(\sum_l \widehat{m}_l(e^b x_1 e^b \otimes ... \otimes e^b x_n e^b)\right) = \\
&= \widehat{d}\left(\sum_{l=0}^{n} \sum_{k=1}^{n-l+1} (-1)^{\cdots}\; e^b x_1 e^b \otimes ... \otimes m(e^b x_k e^b, ..., e^b x_{k+l-1} e^b) \otimes ... \otimes e^b x_n e^b\right) = \\
&= \sum_{j=0}^{n-l+1} \sum_{i=1}^{n-l-j+2} \sum_l \sum_{k=1}^{n-l+1} (-1)^{\cdots}\; e^b x_1 e^b \otimes ... \otimes m(e^b x_i e^b \otimes ... \otimes \\
&\qquad \otimes m(e^b x_k e^b, ..., e^b x_{k+l-1} e^b) \otimes ...) \otimes ... \otimes e^b x_n e^b.
\end{aligned}
\tag{3.100}
$$

The coderivation $\widehat{d^b}$ on $\widehat{B}(C[1])$ acts as follows (on the n-th component)

$$
\begin{aligned}
(\widehat{d^b} \circ \widehat{d^b})(x_1 \otimes ... \otimes x_n) &= \widehat{d^b}(\sum_{l=0}^{n} \widehat{m}^b_l(x_1 \otimes ... \otimes x_n)) = \\
&= \widehat{d^b}(\sum_{l=0}^{n} \sum_{k=1}^{n-l+1} (-1)^{\deg\, x_1 + ... + \deg\, x_{k-1} + k - 1} x_1 \otimes ... \otimes x_{k-1} \otimes \\
&\qquad \otimes m^b_l(x_k, ..., x_{k+l-1}) \otimes x_{k+l} \otimes ... \otimes x_n) = \\
&= \widehat{d^b}(\sum_{l=0}^{n} \sum_{k=1}^{n-l+1} (-1)^{\deg\, x_1 + ... + \deg\, x_{k-1} + k - 1} x_1 \otimes ... \otimes x_{k-1} \otimes \\
&\qquad \otimes m(e^b x_k e^b, ..., e^b x_{k+l-1} e^b) \otimes x_{k+l} \otimes ... \otimes x_n) = \\
&= \sum_{j=0}^{n-l+1} \sum_{i=1}^{n-l-j+2} \sum_{l=0}^{n} \sum_{k=1}^{n-l+1} (-1)^{\cdots} x_1 \otimes ... \otimes x_{i-1} \otimes \\
&\quad \otimes m(e^b x_i e^b \otimes ... \otimes m(e^b x_k e^b, ..., e^b x_{k+l-1} e^b) \otimes ...) ... \otimes x_n \underbrace{=}_{(3.100)} \\
&= 0.
\end{aligned}
\tag{3.101}
$$

Here we deduce that $\widehat{d^b} \circ \widehat{d^b} = 0$ since it is a necessary condition for (3.100) to vanish since $b \equiv 0 \bmod \Lambda^+_{0,nov}$.
Remark that if R-reduction is possible for (C, m) (as in all geometric inspired cases that we regard) it can also be performed for the deformed structure (C, m^b). If $\overline{m}_k$ does not contain $R[e, e^{-1}]$ then neither does $\overline{m}^b_k$.

The second claim about the strictness holds by observing

$$\begin{aligned}
\widehat{d}(e^b) &= \sum_{l=0} \widehat{m}_l(1) + \sum_{l=0}^{1} \widehat{m}_l(b) + \sum_{l=0}^{2} \widehat{m}_l(b \otimes b) + ... = \\
&= \widehat{m}_0(1) + \widehat{m}_0(b) + \widehat{m}_1(b) + \widehat{m}_0(b \otimes b) + \widehat{m}_1(b \otimes b) + \widehat{m}_2(b \otimes b) + ... = \\
&= m_0(1) + m_0(1) \otimes b + (-1)^{\deg\, b+1} b \otimes m_0(1) + ... \underbrace{=}_{\deg\, b=1} \\
&= (1 + b + b \otimes b + ...) \otimes (m_0(1) + m_1(b) + m_2(b,b) + ...) \otimes (1 + b + b \otimes b + ...) = \\
&= e^b m(e^b) e^b = \\
&= e^b m_0^b(1) e^b.
\end{aligned} \tag{3.102}$$

So we conclude that strictness of (C, m^b) (i.e. $m_0^b(1) = 0$) is given if and only if $\widehat{d}(e^b) = 0$. ■

Remark 3.3. *(i) We are deforming the given A_∞-algebra into a strict one in order to find a possible coboundary map.*
For an element b as above with $\widehat{d}(e^b) = 0$ it can now be defined as

$$\boxed{\begin{aligned} &\delta_b(x) := \sum_{l_1,l_0 \geq 0} m_{l_1+l_0+1}(\underbrace{b, ..., b}_{l_1}, x, \underbrace{b, ..., b}_{l_0}) = m(e^b x e^b) = m_1^b(x) \\ &\text{for } x \in B_1(C[1]) = C[1] \end{aligned}} \tag{3.103}$$

As has already been described in the motivation, the A_∞-relation with $m_0^b = 0$ directly leads us to

$$\delta_b \circ \delta_b = m_1^b \circ m_1^b = 0 \tag{3.104}$$

so δ_b is a degree $+1$ map that squares up to 0 that is δ_b a coboundary map.

(ii) The detour via the deformed A_∞-algebra (C, m^b) was necessary to define a map δ_b being a coboundary operation for the complex C.
An alternative approach would be, instead of examining how to deform a given filtered A_∞-algebra (C, m), to define for b_1, b_0 satisfying (3.91)

$$\begin{aligned} &\delta_{b_1,b_0}(x) := \sum_{i,j \geq 0} m_{i+j+1}(\underbrace{b_1, ..., b_1}_{i}, x, \underbrace{b_0, ..., b_0}_{j}) \equiv m(e^{b_1} x e^{b_0}) \\ &\text{for } x \in B_1(C[1]) = C[1] \end{aligned} \tag{3.105}$$

and then consider

$$
\begin{aligned}
&\widehat{d}(e^{b_1}xe^{b_0}) = \\
=&\sum_{i,j\geq 0}\sum_{l=0}^{i+j+1}\widehat{m}_l(\underbrace{b_1\otimes...\otimes b_1}_{i}\otimes x\otimes\underbrace{b_0\otimes...\otimes b_0}_{j}) = \\
=&\sum_{i,j\geq 0}\sum_{l=0}^{i+j+1}\Big(\\
&\quad\sum(-1)^{\cdots}b_1\otimes...\otimes b_1\otimes m_l(b_1,...,b_1)\otimes b_1\otimes...\otimes b_1\otimes x\otimes b_0\otimes...\otimes b_0+ \\
&\quad+\sum(-1)^{\cdots}b_1\otimes...\otimes b_1\otimes m_l(b_1,...,b_1,x,b_0,...,b_0)\otimes b_0\otimes...\otimes b_0+ \\
&\quad+\sum(-1)^{\cdots}b_1\otimes...\otimes b_1\otimes x\otimes b_0\otimes...\otimes b_0\otimes m_l(b_0,...,b_0)\otimes b_0\otimes...\otimes b_0\Big) = \\
=&\widehat{d}(e^{b_1})xe^{b_0}+e^{b_1}m(e^{b_1}xe^{b_0})e^{b_0}+e^{b_1}x\widehat{d}(e^{b_0}) \underbrace{=}_{\text{if } \widehat{d}(e^{b_1})=\widehat{d}(e^{b_0})=0} \\
=&e^{b_1}\delta_{b_1,b_0}(x)e^{b_0}.
\end{aligned}
\tag{3.106}
$$

With the A_∞-relation we thus get

$$
\begin{aligned}
0 &= (\widehat{d}\circ\widehat{d})(e^{b_1}xe^{b_0}) = \widehat{d}(e^{b_1}\delta_{b_1,b_0}(x)e^{b_0}) = \\
&= e^{b_1}(\delta_{b_1,b_0}\circ\delta_{b_1,b_0})(x)e^{b_0}
\end{aligned}
\tag{3.107}
$$

that is, δ_{b_1,b_0} is a coboundary map when $\widehat{d}(e^{b_1})=\widehat{d}(e^{b_0})=0$.
Clearly the concept boils down to

$$\delta_{b_1,b_0}\underbrace{=}_{b_1=b_0=b}\delta_{b,b}\equiv\delta_b \tag{3.108}$$

so the calculation (3.102) is redundant in retrospect, but the more algebraically oriented concepts presented there demonstrate how to construct a strict filtered A_∞-algebra out of a given (not necessarily strict) filtered A_∞-algebra.

Definition 3.7

If there exist elements $b\in(C[1])^0$, $b\equiv 0$ mod $\Lambda^+_{0,nov}$ with

$$\widehat{d}(e^b)=0 \tag{3.109}$$

the underlying A_∞-algebra (C,m) is called (strictly) unobstructed.
The set consisting of those so called (strict) solutions of the Maurer-Cartan equation or equivalently (strict) bounding cochains is denoted by

$$\widehat{\mathcal{M}}_{strict}(C) := \{b\in(C[1])^0 \;\mid\; b\equiv 0 \text{ mod } \Lambda^+_{0,nov},\; \widehat{d}(e^b)=0\}. \tag{3.110}$$

(C,m) is called strictly obstructed for $\widehat{\mathcal{M}}_{strict}(C)=\emptyset$. For

$$b, b_0, b_1 \in \widehat{\mathcal{M}}_{strict}(C) \tag{3.111}$$

the Lagrangian Floer Cohomology for an A_∞-algebra (C,m) is defined by

$$\begin{aligned} HF(C,b;\Lambda_{0,nov}) &:= H(C,\delta_b) \\ HF(C,b_1,b_0;\Lambda_{0,nov}) &:= H(C,\delta_{b_1,b_0}) \end{aligned} \tag{3.112}$$

Be aware that the two following questions are still unanswered and remain to be clarified:

Remark 3.4. *(i) How shall one define the Lagrangian Floer Cohomology in this manner if $\widehat{\mathcal{M}}_{strict}(C)=\emptyset$?*
We refer the reader to the upcoming section 3.2.2 where we try to weaken the requirement

$$\widehat{d}(e^b) \underbrace{=}_{(3.102)} e^b m(e^b) e^b = 0 \quad \textit{and thus} \quad m(e^b)=0. \tag{3.113}$$

There we observe that if (C,m) is unital with unit $\mathbf{e}$, the condition

$$m(e^b) \sim \mathbf{e} \tag{3.114}$$

suffices to define a coboundary map δ_b.
The additionally specification (strict) in the definition above is important when the facts of the next section come into play. We want to be able to distinguish between the strict ($m(e^b)=0$) and the weak ($m(e^b)\sim\mathbf{e}$) bounding cochains b. Remark that we will just omit the pre-indication (strict) when the meaning is clear and just highlight the weak cases.

(ii) How does the defined $HF(C,b;\Lambda_{0,nov})$ respectively $HF(C,b_1,b_0;\Lambda_{0,nov})$ actually depend on the chosen $b,b_0,b_1\in\widehat{\mathcal{M}}_{strict}(C)$?

3.2.2 (Weak) Unobstructedness via the Domain of $\mathfrak{PO}$

The last section posed a quite strict requirement on b to be an element of $\widehat{\mathcal{M}}_{\mathrm{strict}}(C)$. In this section we are now searching for ways to weaken this statement.
We show that elements

$$b \in \widehat{\mathcal{M}}_{\mathrm{weak}}(C) \supset \widehat{\mathcal{M}}_{\mathrm{strict}}(C) \tag{3.115}$$

that is those satisfying

$$m(e^b) \sim \mathbf{e} \tag{3.116}$$

for (C,m) being a unital filtered A_∞-algebra with unit $\mathbf{e}$, are sufficient to define a coboundary map

$$\delta_b : C[1] \to C[1]. \tag{3.117}$$

Further in this context we define a *potential function* $\mathfrak{PO}$ that maps b onto the proportionality factor of (3.116) that is $\mathfrak{PO}|_{\widehat{\mathcal{M}}_{\mathrm{strict}}(C)} = 0$.

Definition 3.8

For a unital A_∞-algebra (C, m) (i.e. $\exists$ unit $\mathbf{e} \in C^0$ satisfying Def. 3.5 (d)) the set of weak solutions of the Maurer-Cartan equation respectively weak bounding cochains is defined as

$$\widehat{\mathcal{M}}_{\mathrm{weak}}(C) := \{b \in (C[1])^0 \ | \ b \equiv 0 \ mod \ \Lambda^{+}_{0,nov}, \ m(e^b) = c \cdot e \cdot \mathbf{e} \quad \text{for } c \in \Lambda^{+(0)}_{0,nov}\} \tag{3.118}$$

for e being the degree 2 generator of $\Lambda_{0,nov}$. Here $\Lambda^{+(0)}_{0,nov}$ denotes the degree 0 part of $\Lambda^{+}_{0,nov}$.
If one can find such an element b the A_∞-algebra (C, m) is called weakly unobstructed.
It is called weakly obstructed for $\widehat{\mathcal{M}}_{\mathrm{weak}}(C) = \emptyset$.

Proposition 3.2

For $b \in \widehat{\mathcal{M}}_{\mathrm{weak}}(C)$ one can define a coboundary operator

$$\begin{aligned} \delta_b^{weak} \equiv \delta_b : C[1] &\to C[1] \\ x &\to \sum_{l_1, l_0 \geq 0} m_{l_1+l_0+1}(\underbrace{b, ..., b}_{l_1}, x, \underbrace{b, ..., b}_{l_0}) \equiv m(e^b x e^b). \end{aligned} \tag{3.119}$$

Proof: Recall the definition 3.45 of

$$\widehat{d} = \sum_{k=0}^{\infty} \widehat{m}_k : \widehat{B}(C[1]) \to \widehat{B}(C[1]). \tag{3.120}$$

A formal way of illustrating this map is

$$\widehat{d} : \widehat{B}(C[1]) \overset{\Delta}{\to} \widehat{B}(C[1]) \otimes \widehat{B}(C[1]) \overset{\mathrm{id}\otimes\Delta}{\longrightarrow} \widehat{B}(C[1])^{\otimes 3} \xrightarrow{(-1)^{\cdots}\mathrm{id}\otimes m \otimes \mathrm{id}} \widehat{B}(C[1]) \tag{3.121}$$

for m as in (3.95). Since $(C, \{m_k\})$ is an A_∞-algebra we have that all components of the image vanish especially the one for $k = 1$ that is in $\widehat{B}_{k=1}(C[1]) = C[1]$. With (3.121) we therefore conclude

$$\widehat{d} \circ \widehat{d} = 0 \Rightarrow m \circ \widehat{d} = 0. \tag{3.122}$$

So we are allowed to continue via

$$
\begin{aligned}
0 =& m(\widehat{d}(e^b x e^b)) = \\
=& m\{\sum_l \widehat{m}_l((1+b+b\otimes b+...)(x)(1+b+b\otimes b+...))\} = \\
=& m\{\sum_{l=0}^{1} \widehat{m}_l(x) + \sum_{l=0}^{2} \widehat{m}_l(b\otimes x) + \sum_{l=0}^{2} \widehat{m}_l(x\otimes b) + ...\} \underbrace{=}_{(*)} \\
=& m\{(1+b+b\otimes b+...)(m_1(x)+m_2(x,b)+m_2(b,x)+m_3(b,x,b)+...)(1+b+...)+ \\
& + ((1+b+b\otimes b+...)(m_0(1)+m_1(b)+m_2(b,b)+...)(1+b+b\otimes b+...)) \\
& \quad x(1+b+b\otimes b+...)+ \\
& + (-1)^{\deg(x)+1}(1+b+b\otimes b+...)x \\
& \quad ((1+b+b\otimes b+...)(m_0(1)+m_1(b)+m_2(b,b)+...)(1+b+b\otimes b+...))\} = \\
=& m(e^b\delta_{b,b}(x)e^b) + m(\widehat{d}(e^b)xe^b) + (-1)^{\deg(x)+1} m(e^b x \widehat{d}(e^b)) \underbrace{=}_{(3.102)} \\
=& \sum_l m_{2l+1}(\underbrace{b,...,b}_{l}, \delta_{b,b}(x), \underbrace{b,...,b}_{l}) + m(e^b m(e^b) e^b x e^b)+ \\
& + (-1)^{\deg(x)+1} m(e^b x e^b m(e^b) e^b) \underbrace{=}_{(3.118)} \\
=& (\delta_{b,b}\circ\delta_{b,b})(x) + ce\cdot \underbrace{m(e^b \mathbf{e} e^b x e^b)}_{=m_2(\mathbf{e},x)} + (-1)^{\deg(x)+1} ce \cdot \underbrace{m(e^b x e^b \mathbf{e} e^b)}_{=m_2(x,\mathbf{e})} \underbrace{=}_{\text{Def.}(3.5)(d)} \\
=& (\delta_{b,b}\circ\delta_{b,b})(x) + ce\cdot x + (-1)^1 ce\cdot x = \\
=& (\delta_{b,b}\circ\delta_{b,b})(x).
\end{aligned}
$$

Remark that for $(*)$ we reordered the summands and that according to (3.42) for the 2nd and the 3rd part of the upcoming terms one has

$$
\begin{aligned}
& (-1)^{n(\deg(x)+1)+(k-1)(\overbrace{\deg(b)}^{=1}+1)} = \\
=& (-1)^{n(\deg(x)+1)} = \begin{cases} 1, & n=0 \text{ for } 2^{\text{nd}} \text{ part} \\ (-1)^{\deg(x)+1}, & n=1 \text{ for } 3^{\text{rd}} \text{ part} \end{cases}
\end{aligned}
\tag{3.123}
$$

■

Definition 3.9

For

$$b \in \widehat{\mathcal{M}}_{weak}(C) \tag{3.124}$$

the Lagrangian Floer Cohomology for an unital filtered A_∞-algebra (C, m) is defined by

$$HF(C, b; \Lambda_{0,nov}) := H(C, \delta_b) \tag{3.125}$$

The headline of this section advises us to examine the domain of a *potential function* $\mathfrak{PO}$. Here in this setup it is defined as a mapping

$$\boxed{\begin{aligned}&\mathfrak{PO} : \widehat{\mathcal{M}}_{\text{weak}}(C) \to \Lambda^{+(0)}_{0,nov}\\ &\mathfrak{PO}(b) \text{ defined by } m(e^b) = \mathfrak{PO}(b)e\mathbf{e}\end{aligned}} \tag{3.126}$$

One has $\widehat{\mathcal{M}}_{\text{strict}}(C) \subset \widehat{\mathcal{M}}_{\text{weak}}(C)$ and therefore for b with $\mathfrak{PO} \equiv 0$ the weak unobstructed case can be strenghtened to the results of section 3.2.1.

A natural question arises about Maurer-Cartan solutions in the context of this and the preceding section.
What happens when applying unital (see (3.73)) A_∞-homomorphisms to them?

The following argumentation works likewise for unobstructed respectively weakly unobstructed A_∞-algebras. So we neglect the indication strict and weak in the following.
Our goal is to show that for a given unital A_∞-homomorphism

$$f : (C^1, m^1) \to (C^2, m^2) \tag{3.127}$$

it is possible to define an induced map

$$f_* : \underbrace{(C_1[1])^0_{\text{mod}}}_{\supset\widehat{\mathcal{M}}(C_1)} := \{x \in (C[1])^0 \mid x \equiv 0 \bmod \Lambda^+_{0,nov}\} \to \underbrace{(C_2[1])^0_{\text{mod}}}_{\supset\widehat{\mathcal{M}}(C_2)} \tag{3.128}$$

that preserves the property of being a Maurer-Cartan solution that is we have

$$f_* : \widehat{\mathcal{M}}(C_1) \to \widehat{\mathcal{M}}(C_2). \tag{3.129}$$

Even a stronger fact can be proven, namely that the commutation relation

$$\mathfrak{PO}_2 \circ f_* = \mathfrak{PO}_1 \tag{3.130}$$

is satisfied.

The map $f_* : (C_1[1])^0_{\text{mod}} \to (C_2[1])^0_{\text{mod}}$ shall be defined as

$$f_*(x) := f(e^x) = f_0(1) + f_1(x) + f_2(x,x) + \dots\ . \tag{3.131}$$

The image $\text{im}(f_*)$ is really contained in 0-th part $(C_2[1])^0$ since f is assumed to be of degree 0 (see (3.65)). In (3.66) we further required that the energy filtration is preserved by f. This thus provides the additional information that

$$f_*(x) \in (C_2[1])^0_{\text{mod}}. \tag{3.132}$$

Lemma 3.4

A unital filtered A_∞-homomorphism f between unital filtered A_∞-algebras (C_1, m^1) and (C_2, m^2) (with units $\mathbf{e}_1$ resp. $\mathbf{e}_2 = f_1(\mathbf{e}_1)$) induces a map

$$\begin{aligned} f_* : \widehat{\mathcal{M}}(C_1) &\to \widehat{\mathcal{M}}(C_2)\\ b &\mapsto f(e^b)\end{aligned} \tag{3.133}$$

Further the commutation relation

$$\mathfrak{PO}_2 \circ f_* = \mathfrak{PO}_1 \tag{3.134}$$

is satisfied for

$$\mathfrak{PO}_i : \widehat{\mathcal{M}}_{weak}(C_i) \to \Lambda^{+(0)}_{0,nov} \quad (i = 1, 2) \tag{3.135}$$

being the corresponding potential functions.

Proof: Take an element $b \in \widehat{\mathcal{M}}(C_1)$, this means that either

$$\widehat{d}^1(e^b) \underbrace{=}_{(3.102)} e^b m^1(e^b) e^b = 0 \Rightarrow \mathfrak{PO}_1(b) = 0 \tag{3.136}$$

in the strict or

$$m^1(e^b) = c e \mathbf{e}_1 \Rightarrow \mathfrak{PO}_1(b) = c \tag{3.137}$$

for the weak case. According to the definition of $f_*(b) = f(e^b)$ we have to check whether

$$\widehat{d}^2(e^{f(e^b)}) \underbrace{=}_{\text{proof analog. to (3.102)}} e^{f(e^b)} m^2(e^{f(e^b)}) e^{f(e^b)} = 0 \Rightarrow \mathfrak{PO}_2(f(e^b)) = 0 \tag{3.138}$$

respectively

$$m^2(e^{f(e^b)}) = c' e \mathbf{e}_2 \Rightarrow \mathfrak{PO}_2(f(e^b)) = c' \tag{3.139}$$

holds.
The assertion about the commutation of f_* with $\mathfrak{PO}_i$ holds then trivially in the strict case since

$$\mathfrak{PO}_i \equiv 0 \quad (i = 1, 2). \tag{3.140}$$

In the weak case we are done if we can show that

$$\mathfrak{PO}_1(b) = c = c' = \mathfrak{PO}_2(f_*(b)). \tag{3.141}$$

To get a better insight on what is happening let us analyse the term $e^{f(e^b)}$ more properly:

$$
\begin{aligned}
e^{f(e^b)} = & \\
&= 1 + f(e^b) + f(e^b) \otimes f(e^b) + ... = \\
&= 1 + (f_0(1) + f_1(b) + ...) + (f_0(1) + f_1(b) + ...) \otimes (f_0(1) + f_1(b) + ...) + ... = \\
&= 1 + f_0(1) + f_0(1) \otimes f_0(1) + ... + \\
&\quad + f_1(b) + f_1(b) \otimes f_0(1) + f_0(1) \otimes f_1(b) + f_0(1) \otimes f_0(1) \otimes f_1(b) + ... + \\
&\quad + f_2(b \otimes b) + ... = \\
&= 1 + f_0(1) + f_0(1) \otimes f_0(1) + ... + \\
&\quad + \sum_{0 \le k_1 \le ... \le k_n \le 1} f_{k_1}(\underbrace{b, ..., b}_{k_1}) \otimes ... \otimes f_{1-k_n}(\underbrace{b, ..., b}_{1-k_n}) + \\
&\quad + \sum_{0 \le k_1 \le ... \le k_n \le 2} f_{k_1}(\underbrace{b, ..., b}_{k_1}) \otimes ... \otimes f_{2-k_n}(\underbrace{b, ..., b}_{2-k_n}) + \\
&\quad + ... \underbrace{=}_{(3.68),(3.69)} \\
&= \widehat{f}(1) + \widehat{f}(b) + \widehat{f}(b \otimes b) + ... = \\
&= \widehat{f}(1 + b + b \otimes b + ...) = \\
&= \widehat{f}(e^b)
\end{aligned}
\tag{3.142}
$$

The equation $e^{f(e^b)} = \widehat{f}(e^b)$ allows to continue

$$
\begin{aligned}
e^{f(e^b)} m^2(e^{f(e^b)}) e^{f(e^b)} &= \widehat{d}^2(e^{f(e^b)}) = \widehat{d}^2(\widehat{f}(e^b)) \underbrace{=}_{(3.71)} \\
&= \widehat{f}(\widehat{d}^1(e^b)) = \widehat{f}(e^b m^1(e^b) e^b) = \\
&= \widehat{f}(e^b c e \mathbf{e}_1 e^b) = \\
&= c e \widehat{f}((1 + b + b \otimes b + ...) \mathbf{e}_1 (1 + b + b \otimes b + ...)) \underbrace{=}_{(3.73)} \\
&= c e \Big[(1 + f(e^b) + f(e^b) \otimes f(e^b) + ...) f_1(\mathbf{e}_1) \\
&\qquad\qquad (1 + f(e^b) + f(e^b) \otimes f(e^b) + ...) \Big] = \\
&= e^{f(e^b)} (c e f_1(\mathbf{e}_1)) e^{f(e^b)} = \\
&= e^{f(e^b)} (c e \mathbf{e}_2) e^{f(e^b)}.
\end{aligned}
\tag{3.143}
$$

When comparing both hand sides one gets

$$
m^2(e^{f(e^b)}) = c e \mathbf{e}_2 \tag{3.144}
$$

and thus $c = c'$. This forms the end of the proof and at the same time finishes our at some parts geometrically inspired review about A_∞-algebras focusing on the construction of coboundary operators. ∎

Chapter 4

Fundamentals of Kuranishi Structures

The preceding chapter introduces the useful framework of A_∞-algebras. It is the underlying concept out of which one wants to construct a Lagrangian Floer cohomology theory under certain conditions posed on the Lagrangian submanifold L. Another necessary tool, so called *Kuranishi structures* become helpful for describing concrete geometric situations, like in our case the behavior of Lagrangian submanifolds $L \subset M$, in an A_∞ fashion. There are several ways of how to face these assignments. All of these methods have an important ingredient in common. One has to find a way of how the moduli space $\mathcal{M}$ of pseudo-holomorphic curves attaching the Lagrangians can be treated in a transparent fashion.
As in most parts of this text we will follow the ideas of K. Fukaya et al.. See [FOOO1] for a treatment of the general theory respectively [FOOO2] for how it is specialized to the situation of toric symplectic manifolds.
In standard Floer theory a proper handling of the moduli space of pseudo-holomophic curves is necessary in order to find an appropriate boundary operator. In our situation we are confronted with similar difficulties, namely transversality and orientation issues of the moduli space. Unfortunately emerging complicacies seem to be even more hard to handle. For example moduli spaces $\mathcal{M}_{l+1}(\beta)$ for different numbers l of marked points and homology types β have to be considered at the same time, that is boundaries arise as fiber products of other moduli spaces (see e.g. (5.91)).
In this chapter we remain to stay on a very formal level. This is due to the fact that the concept of Kuranishi structures find applications in different areas of mathematics (see e.g. [J]) and physics (see e.g. [ChRu]). So let us give a short outline on what will be done here.
We first summarize some main facts about the general buildup of K. structures. A discussion follows, aiming to clarify the meaning of endowing K. structures with tangent bundles and what is meant by the notion of orientability in the context.
The goal of this thesis is to define and actually compute Lagrangian Floer cohmology for (at least some kind of) toric symplectic manifolds (see also chapter 5.2 for a short recap of these notion). In this context we discuss the meaning of group actions on K. structures or more concretely, what is meant by T^n-equivariance (in the sense of K. structures). The theory allows (and we actually need) to speak of so called

multisections $\mathfrak{s}$ of the corresponding *obstruction bundles* and we then can ask for transversality. As usually we then consider the zero level sets of these (transversal) multisections. These thereof arising *pertubed moduli spaces* $\mathcal{M}^{\mathfrak{s}}$ will be the basis of how we attend to define the A_∞-homomorphisms

$$m_l : B_l(C[1]) \to C[1] \; . \tag{4.1}$$

For them there are at least two ways of how one can introduce them. In general this is done on the chain level by using *virtual chains* (see section 3.4. and 3.5. of [FOOO1] for details).
Since we aim to actually compute some of the maps m_l (see e.g. section 5.5), for our toric setup (see [FOOO2] for details) it is more useful to follow a slight different approach. Working on the cochain level, we aim to transport harmonic forms from *source* (l copies of L) to *target* Lagrangians (1 copy of L).
In order to really see that the maps m_l fulfill the required A_∞-relation (3.63), we deduce that this "transportation" of forms can be composed in some way and we even can write down an analogue of Stokes' theorem for it. This directly leads us to chapter 5, where we try to apply these newly developed concepts in order to describe the setup $L \subset M$ in an A_∞-algebra fashion.

4.1 Good coordinate systems

At the beginning of the modern study of the geometry of topologial spaces one wanted to get a local feeling for them. If possible this is done by covering them with charts. Homeomorphisms to "nice" spaces ($\mathbb{R}^n$, $\mathbb{C}^n$, ...) allow to use well established mathematics. Here we try to do something similar with our moduli spaces although it is a bit less transparent. We start by clarifying the meaning of neighborhoods and coordinate changes in the sense of K. structures.

Definition 4.1

Let $\mathcal{M}$ be a compact metrizable space. A covering with Kuranishi charts is an assignment of a quintuple called Kuranishi neighborhood $(V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p})$ $(\alpha_p \in I)$ for all $p \in \mathcal{M}$ with the following properties:

(i) V_{α_p} *being a finite dimensional smooth manifold;*

(ii) E_{α_p} *being a finite dimensional vector space over* $\mathbb{R}$*;*

(iii) Γ_{α_p} *being a finite group with an effective group action on* V_{α_p} *(that is* $\bigcap\limits_{o_p \in V_{\alpha_p}} \{g \in \Gamma_{\alpha_p} \mid g.o_p = o_p\} = \{e\}$*) and a linear representation on* E_{α_p}*;*

(iv) s_{α_p} *being a* Γ_{α_p} *equivariant section of the vector bundle*

$$E_{\alpha_p} \times V_{\alpha_p} \xrightarrow{pr_{\alpha_p}} V_{\alpha_p} \tag{4.2}$$

for pr_{α_p} *being the* Γ_{α_p} *equivariant projection onto the second factor;*

(v) $s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}$ *is homeomorphic to* $im(\psi_{\alpha_p}) \subset \mathcal{M}$ *being a neighborhood of* $p \in \mathcal{M}$ *via*

$$\psi_{\alpha_p} : s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p} \to \mathcal{M} \; ; \tag{4.3}$$

We call s_{α_p} *a Kuranishi map of the corresponding obstruction bundle*

$$E_{\alpha_p} \times V_{\alpha_p} \longrightarrow V_{\alpha_p} \; . \tag{4.4}$$

One further assumes that the set of points

$$\{o_p \in V_{\alpha_p} \mid s_{\alpha_p}(o_p) = 0 \; ; \; \psi_{\alpha_p}([o_p]) = p\} \tag{4.5}$$

is fixed by Γ_{α_p}.

Again reinterpreting the classical approach of how to get insight to the structure of topological spaces, we now have to find a way to change between different coordinates. We show why one has to worry about the fact whether a coordinate system shall be seen as a good one and what the meaning of 'good' in this context actually is.

Definition 4.2

Assume Kuranishi charts $(V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p})$, $(V_{\alpha_q}, E_{\alpha_q}, \Gamma_{\alpha_q}, \psi_{\alpha_q}, s_{\alpha_q})$ *on* $\mathcal{M}$ *are given with* $q \in \psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p})$.
Let further V_{α_p,α_q} *be an* Γ_{α_q} *invariant* ($\Gamma_{\alpha_q}.V_{\alpha_p,\alpha_q} = V_{\alpha_p,\alpha_q}$) *open neighborhood of* o_p *(fulfilling (4.5)) in* V_{α_q}.
A coordinate change is given by a triple $(\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}, \widehat{\phi}_{\alpha_p,\alpha_q}, \phi_{\alpha_p,\alpha_q})$ *if it fulfills:*

(i) $\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}$ *being an injective group homomorphism* $\Gamma_{\alpha_q} \to \Gamma_{\alpha_p}$;

(ii)

$$E_q \times V_{\alpha_p,\alpha_q} \xrightarrow{(\widehat{\phi}_{\alpha_p,\alpha_q}, \phi_{\alpha_p,\alpha_q})} E_p \times V_{\alpha_p} \tag{4.6}$$

being a $\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}$ *equivariant embedding covering the smooth* $\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}$ *equivariant embedding* $V_{\alpha_p,\alpha_q} \xrightarrow{\phi_{\alpha_p,\alpha_q}} V_{\alpha_p}$;

(iii) *the well defined restriction of* $\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}$ *to*

$$(\Gamma_{\alpha_q})_{o_q} \to (\Gamma_{\alpha_p})_{\phi_{\alpha_p,\alpha_q}(o_q)}, \tag{4.7}$$

for $(\Gamma_{\alpha_q})_{o_q}$ *being the stabilizer of* $o_q \in V_{\alpha_p,\alpha_q}$, *is an isomorphism;*

(iv) ϕ_{α_p,α_q} *induces an injective map* $\underline{\phi}_{\alpha_p,\alpha_q} : V_{\alpha_p,\alpha_q}/\Gamma_{\alpha_q} \to V_{\alpha_p}/\Gamma_{\alpha_p}$;

(v) $\widehat{\phi}_{\alpha_p,\alpha_q} \circ s_{\alpha_q} \mid_{V_{\alpha_p,\alpha_q}} = s_{\alpha_p} \circ \phi_{\alpha_p,\alpha_q}$

(vi) $\psi_{\alpha_q} \mid_{(s_{\alpha_q}^{-1}(0) \cap V_{\alpha_p,\alpha_q})/\Gamma_{\alpha_q}} = \psi_{\alpha_p} \circ \underline{\phi}_{\alpha_p,\alpha_q} \mid_{(s_{\alpha_q}^{-1}(0) \cap V_{\alpha_p,\alpha_q})/\Gamma_{\alpha_q}}$

A Kuranishi structure for $\mathcal{M}$ is then the allocation of a Kuranishi neighborhood for all $p \in \mathcal{M}$ and appropriate coordinate changes for all $q \in \psi_{\alpha_p}(s^{-1}_{\alpha_p}(0)/\Gamma_{\alpha_p})$ fulfilling:

(i) The virtual dimension

$$\text{vir.dim}_p(\mathcal{M}) := \dim V_{\alpha_p} - \dim E_{\alpha_p} \equiv \text{vir.dim}\,(\mathcal{M}) \tag{4.8}$$

does not depend on p;

(ii) For $r \in \psi_{\alpha_q}((V_{\alpha_p,\alpha_q} \cap s^{-1}_{\alpha_q}(0))/\Gamma_{\alpha_q})$ one gets

$$\underline{\phi}_{\alpha_p,\alpha_q} \circ \underline{\phi}_{\alpha_q,\alpha_r} = \underline{\phi}_{\alpha_p,\alpha_r}; \tag{4.9}$$

Remark 4.1. *Let us illustrate point (iii) a bit more accurate. It is clear that we have*

$$\gamma \in (\Gamma_{\alpha_q})_{o_q} \subset \Gamma_{\alpha_q} \;\Rightarrow\; \widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(\gamma) \in (\Gamma_{\alpha_p})_{\phi_{\alpha_p,\alpha_q}(o_q)} \subset \Gamma_{\alpha_p} \tag{4.10}$$

for $o_q \in V_{\alpha_p,\alpha_q}$ since $\phi_{\alpha_p,\alpha_q}(o_q) = \phi_{\alpha_p,\alpha_q}(\gamma.o_q) = \widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(\gamma).\phi_{\alpha_p,\alpha_q}(o_q)$. A similar argumentation using the isomorphism property of (iii) and the injectivity of (iv) provides us an alternative formulation of (iii):

$$\text{(iii)'}\quad (\gamma.\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q})) \cap (\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q})) \neq \emptyset \quad \text{for } \gamma \in \Gamma_{\alpha_p} \tag{4.11a}$$

$$\Rightarrow \gamma \in \widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(\Gamma_{\alpha_q}) \tag{4.11b}$$

This fact follows easily by the following observation

$$\begin{aligned} (4.11a) &\Rightarrow \exists\; x, y \in V_{\alpha_p,\alpha_q};\; \gamma \in \Gamma_{\alpha_q} \\ &\qquad s.th.\; \phi_{\alpha_p,\alpha_q}(x) = \gamma.\phi_{\alpha_p,\alpha_q}(y) \\ &\Rightarrow [\phi_{\alpha_p,\alpha_q}(x)] = [\phi_{\alpha_p,\alpha_q}(y)] \;\in V_{\alpha_p}/\Gamma_{\alpha_p}\;. \end{aligned} \tag{4.11c}$$

By the injectivity of $\underline{\phi}_{\alpha_p,\alpha_q} : V_{\alpha_p,\alpha_q}/\Gamma_{\alpha_q} \to V_{\alpha_p}/\Gamma_{\alpha_p}$ we therefore can deduce

$$\begin{aligned} &[x] = [y] \quad \text{that is} \quad x = \beta.y \quad \text{for} \quad \beta \in \Gamma_{\alpha_q} \\ &\Rightarrow \phi_{\alpha_p,\alpha_q}(y) = \gamma^{-1}.(\phi_{\alpha_p,\alpha_q}(\beta.y)) \underbrace{=}_{(ii)} (\gamma^{-1} \cdot \widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(\beta)).(\phi_{\alpha_p,\alpha_q}(y)) \\ &\Rightarrow \gamma^{-1} \cdot \widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(\beta) \in (\Gamma_{\alpha_p})_{\phi_{\alpha_p,\alpha_q}(y)}\;. \end{aligned} \tag{4.11d}$$

Due to the isomorphism property (iii) one can find an group element $g \in (\Gamma_{\alpha_q})_y$ with $\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(g) = \gamma^{-1} \cdot \widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(\beta)$ and we thus conclude

$$\gamma = \underbrace{\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(\beta) \cdot (\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(g))^{-1}}_{\in\, \widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}(\Gamma_{\alpha_q})}\;. \tag{4.11e}$$

After clarifying what is meant by K.- charts and K. coordinate changes, we aim to clarify why "good" coordinate systems are necessary to be considered in this context. The problem arises since we just required that vir.dim($\mathcal{M}$) is independent of the underlying $p \in \mathcal{M}$. So coordinate changes do not in general have to exist since we do not exclude $\dim V_{\alpha_p} \neq \dim V_{\alpha_q}$ for $p \neq q$.
We discuss a possibility of how the K. neighborhoods can be ordered in a clever way, such that the coordinate changes exist at least in one direction. That such an ordering can always be found is guaranteed by the proceeding lemma.

Lemma 4.1

The index set I of a space with Kuranishi structure carries a partial order $<$. For $\alpha_p, \alpha_q \in I$ with $\psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) \cap \psi_{\alpha_q}(s_{\alpha_q}^{-1}(0)/\Gamma_{\alpha_q}) \neq \emptyset$ one either has

$$\alpha_p \leq \alpha_q \quad \text{or} \quad \alpha_q \leq \alpha_p \ .$$

In such a situation (wlog. $\alpha_q \leq \alpha_p$) one can always find $(V_{\alpha_p,\alpha_q}, \widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}, \widehat{\phi}_{\alpha_p,\alpha_q}, \phi_{\alpha_p,\alpha_q})$ satisfying the properties stated in Definition (4.2). Remark that (4.9) then holds for

$$\psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) \cap \psi_{\alpha_q}(s_{\alpha_q}^{-1}(0)/\Gamma_{\alpha_q}) \cap \psi_{\alpha_r}(s_{\alpha_r}^{-1}(0)/\Gamma_{\alpha_r}) \neq \emptyset$$

with $\alpha_r \leq \alpha_p \leq \alpha_p$.
In addition one has

(i) $\psi_{\alpha_q}^{-1}(\psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) \cap \psi_{\alpha_q}(s_{\alpha_q}^{-1}(0)/\Gamma_{\alpha_q})) \subset V_{\alpha_p,\alpha_q}/\Gamma_{\alpha_q}$;

(ii) $\bigcup_{\alpha_p \in I} \psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) = \mathcal{M}$

A coordinate system $(\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}, \widehat{\phi}_{\alpha_p,\alpha_q}, \phi_{\alpha_p,\alpha_q})$ fulfilling these properties is specified to be a good coordinate system.

Proof: [FO] ■

We already remarked our goal, namely to equip the moduli space of pseudo-holomorphic curves with a Kuranishi structure. In this context the evaluation map

$$ev = (ev_1, ..., ev_l, ev_0) \tag{4.12}$$

from $\mathcal{M}_{l+1}(\beta)$ to the Lagrangian L^{k+1} will play an important role in order to define the homomorphisms $\{m_l\}_{l \geq 0}$ for the A_∞-algebra structure. Therefore we want to find ways of how maps from spaces with K. structure into smooth manifolds can be treated in a meaningful fashion.

Definition 4.3

Assume for a space $\mathcal{M}$ with Kuranishi structure and for all $p \in \mathcal{M}$ one has Γ_{α_p} invariant, continous maps f_{α_p} mapping V_{α_p} into a topological manifold M. They are said to be strongly continous if one has

$$f_{\alpha_p} \circ \phi_{\alpha_p,\alpha_q} = f_{\alpha_q} \tag{4.13}$$

when restricting to $V_{\alpha_p,\alpha_q} \subset V_{\alpha_q}$.

A simply observation shows that one always can uniquely construct a continuous map

$$f : \mathcal{M} \to M \tag{4.14}$$

out of a family $\{f_{\alpha_p}\}$ of strongly continous maps. Remark that we have

$$\bigcup_{\alpha_p \in I} \psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) = \mathcal{M} \ . \tag{4.15}$$

Then it is defined via

$$f(x) := (f_{\alpha_p} \circ \pi_{\alpha_p}^{-1} \circ \psi_{\alpha_p}^{-1})(x) \quad \text{for} \quad x \in \psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) \tag{4.16}$$

and π_{α_q} being the projection $V_{\alpha_p} \to V_{\alpha_p}/\Gamma_{\alpha_p}$. Due to the required invariance of f_{α_p} it does not matter which representative of the equivalence class of the quotient space $V_{\alpha_p}/\Gamma_{\alpha_p}$ is chosen. Remark that $f(x)$ is already continuous on $\psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p})$ since ψ_{α_p} is a homeomorphism onto its image and $V_{\alpha_p}/\Gamma_{\alpha_p}$ carries the quotient topology. Because of (4.13) one has

$$\begin{aligned}
(f_{\alpha_q} \circ \pi_{\alpha_q}^{-1} \circ \psi_{\alpha_q}^{-1})(x) = & \\
&= (f_{\alpha_p} \circ \phi_{\alpha_p,\alpha_q} \circ \pi_{\alpha_q}^{-1} \circ \psi_{\alpha_q}^{-1})(x) \underbrace{=}_{\text{Def.}(4.2)\ (\text{iv})} \\
&= (f_{\alpha_p} \circ \pi_{\alpha_p}^{-1} \circ \underline{\phi}_{\alpha_p,\alpha_q} \circ \psi_{\alpha_q}^{-1})(x) \underbrace{=}_{\text{Def.}(4.2)\ (\text{vi})} \\
&= (f_{\alpha_p} \circ \pi_{\alpha_p}^{-1} \circ \psi_{\alpha_p}^{-1})(x)
\end{aligned} \tag{4.17}$$

This implies that f is well defined and continuous on the overlaps (wlog. $\alpha_q \leq \alpha_p$) for

$$x \in \psi_{\alpha_p}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) \cap \psi_{\alpha_q}(s_{\alpha_q}^{-1}(0)/\Gamma_{\alpha_q}) \tag{4.18}$$

that is

$$\psi_{\alpha_q}^{-1}(x) \in (s_{\alpha_q}^{-1}(0) \cap V_{\alpha_p,\alpha_q})/\Gamma_{\alpha_q} \quad \text{and} \quad \psi_{\alpha_p}^{-1}(x) \in s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p} \ . \tag{4.19}$$

If a function $f : \mathcal{M} \to M$ arising in a way as described here we additionally call it *strongly continous*.

Definition 4.4

If the manifold M is smooth then strongly continuous maps $f : \mathcal{M} \to M$ are said to be weakly smooth respectively weakly submersive if the maps

$$f_{\alpha_p} : V_{\alpha_p} \to M \tag{4.20}$$

are smooth respectively submersive for all $p \in \mathcal{M}$.

Here one should be aware of the following fact. Assume two Kuranishi neighborhoods for $p, q \in \mathcal{M}$ are given with

$$x = \psi_{\alpha_q}([x_{\alpha_q}]) = \psi_{\alpha_p}([x_{\alpha_p}]) \tag{4.21}$$

that is

$$x \in \psi_{\alpha_q}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) \cap \psi_{\alpha_q}(s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}) \subset M \ . \tag{4.22}$$

We can consider the differentials of the sections s_{α_p}, s_{α_q} of the obstruction bundle at the point x_{α_p} respectively x_{α_q}, that is

$$d_{x_{\alpha_p}} s_{\alpha_p} : T_{x_{\alpha_p}} V_{\alpha_p} \longrightarrow (E_{\alpha_p})_{x_{\alpha_p}} \times V_{\alpha_p} \ ; \ d_{x_{\alpha_q}} s_{\alpha_q} : T_{x_{\alpha_q}} V_{\alpha_q} \longrightarrow (E_{\alpha_q})_{x_{\alpha_q}} \times V_{\alpha_q} \ . \tag{4.23}$$

It would be nice to find a way of how both can be linked in some way. First consider $\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q})$ as an embedded submanifold of V_{α_p}. We abbreviate $\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q})$ simple as V_{α_p,α_q} if the meaning is clear, that it sits in V_{α_p} as an embedded submanifold. The restriction of ds_{α_p} to the normal bundle $NV_{\alpha_p,\alpha_q} = TV_{\alpha_p}/TV_{\alpha_p,\alpha_q}$ yields a map

$$d^{\text{fibre}} s_{\alpha_p} : NV_{\alpha_p,\alpha_q} \longrightarrow E_{\alpha_p} \times NV_{\alpha_p,\alpha_q} \ . \tag{4.24}$$

As usually the exponential map can be used to identify neighborhoods of V_{α_p,α_q} in V_{α_p} with neighborhoods of the zero section in the normal bundle. So we get a $\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}$ equivariant bundle homomorphism

$$d^{\text{fibre}} s_{\alpha_p} : NV_{\alpha_p,\alpha_q} \longrightarrow E_{\alpha_p} \times V_{\alpha_p,\alpha_q} \ . \tag{4.25}$$

Definition 4.5

A space $\mathcal{M}$ equiped with a Kuranishi structure has a tangent bundle (in the sense of K. structures) if $d^{\text{fibre}} s_{\alpha_p}$ induces a bundle isomorphism between the Γ_{α_q} equivariant bundles over $\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q})$, namely

$$TV_{\alpha_p}/T\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q}) \cong \tag{4.26}$$

$$\cong E_{\alpha_p} \times \phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q})/(\widehat{\phi}_{\alpha_p,\alpha_q}, \phi_{\alpha_p,\alpha_q})(E_{\alpha_q} \times V_{\alpha_p,\alpha_q}) \ .$$

$\mathcal{M}$ is additionally called oriented (in the sense of K. structures) if for all $p \in \mathcal{M}$ the manifolds V_{α_p} and the bundles $E_{\alpha_p} \times V_{\alpha_p}$ are oriented and the group action of Γ_{α_p} and the fiber derivative $d^{\text{fibre}} s_{\alpha_p}$ are preserving the orientation.

4.2 T^n equivariant Kuranishi structures for $\mathcal{M}_{l+1}(\beta)$

According to Fukaya et al. there is a detailed definition of how to characterize group actions $G \hookrightarrow \mathcal{M}$ in the sense of Kuranishi structures (see e.g. [FOOO1] and [FOOO2]). There the authors prove that for finite groups the K. structure can in general be devolved to the quotient space $\mathcal{M}/G$. Since we just consider the action of the n-tori T^n at a later stage of progress, we are content with a slightly simplified version of the general definition of group actions on K. structures (see e.g.[FOOO2]). In the following we always assume that underlying space $\mathcal{M}$ is already equipped with an K. structure and a T^n action on $\mathcal{M}$ is given.

Definition 4.6

The Kuranishi structure of $\mathcal{M}$ is called T^n equivariant if the following holds for K. neighborhoods $(V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p})$ and the corresponding coordinate changes $(\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}, \widehat{\phi}_{\alpha_p,\alpha_q}, \phi_{\alpha_p,\alpha_q})$ for all $p, q \in \mathcal{M}$:

(i) V_{α_p} carries a T^n action commuting with the Γ_{α_p} action;

(ii) the vector bundle $E_{\alpha_p} \times V_{\alpha_p} \to V_{\alpha_p}$ is T^n equivariant;

(iii) $s_{\alpha_p} : V_{\alpha_p} \to E_{\alpha_p} \times V_{\alpha_p}$ is a T^n equivariant section;

(iv) $\widehat{\phi}_{\alpha_p,\alpha_q}$, ϕ_{α_p,α_q} are T^n equivariant;

(v) $\psi_{\alpha_p} : s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p} \to \mathcal{M}$ is T^n equivariant;

Strongly continuous maps $f : \mathcal{M} \to M$ are called T^n equivariant if the corresponding maps $f_{\alpha_p} : V_{\alpha_p} \to M$ are T^n equivariant for all $p \in \mathcal{M}$.

For condition (v) we remark that the T^n action on $s_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p}$ is well defined since s_{α_p} is T^n equivariant and the actions of T^n and Γ_{α_p} are assumed to commute with each other.

A simple observations shows that $\mathcal{M}/T^n$ also carries a natural induced K. structure if one has the T^n action to be free on each K. neighborhood. It can be derived as follows for all $p \in \mathcal{M}$:
Take a K. chart $(V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p})$. V_{α_p}/T^n is also a smooth manifold since T^n acts free on V_{α_p}.
Due to the required T^n equivariance of $E_{\alpha_p} \times V_{\alpha_p} \to V_{\alpha_p}$ we are allowed to consider the orbit bundle

$$(E_{\alpha_p} \times V_{\alpha_p})/T^n \to V_{\alpha_p}/T^n \tag{4.27}$$

as a vector bundle with finite dimensional real vector spaces

$$((E_{\alpha_p} \times V_{\alpha_p})/T^n)_{[x_p]} \quad \text{for} \quad [x_p] \in V_{\alpha_p}/T^n \tag{4.28}$$

as its fibers.
Γ_{α_p} can just be taken to be the corresponding group acting on the vector bundle

$$(E_{\alpha_p} \times V_{\alpha_p})/T^n \to V_{\alpha_p}/T^n \tag{4.29}$$

since we assumed that the group actions of T^n and Γ_{α_p} commute with each other.
Similarly the T^n equivariance of the sections s_{α_p} allows to define Kuranishi maps

$$\overline{s}_{\alpha_p} := \pi_{E_{\alpha_p}} \circ s_{\alpha_p} \circ \pi_{V_{\alpha_p}}^{-1} : V_{\alpha_p}/T^n \to (E_{\alpha_p} \times V_{\alpha_p})/T^n \tag{4.30}$$

for $\pi_{E_{\alpha_p}}, \pi_{V_{\alpha_p}}$ being the projection of $E_{\alpha_p} \times V_{\alpha_p}$ respectively V_{α_p} to the corresponding orbit space arising by the T^n action.
ψ_{α_p} can be projected to map $\overline{\psi}_{\alpha_p}$ between the orbit spaces

$$\overline{\psi}_{\alpha_p} : (\overline{s}_{\alpha_p}^{-1}(0)/\Gamma_{\alpha_p})/T^n \to \mathcal{M}/T^n \tag{4.31}$$

since we assumed T^n equivariance for ψ_{α_p}.
In summary the devolved K. structure for $\mathcal{M}/T^n$ is given by

$$(V_{\alpha_p}/T^n, E_{\alpha_p}/T^n, \Gamma_{\alpha_p}, \overline{\psi}_{\alpha_p}, \overline{s}_{\alpha_p}) \ . \tag{4.32}$$

The coordinate change for the K. structure of $\mathcal{M}/T^n$ is defined by

$$(\overline{\widehat{\phi}}_{\alpha_p,\alpha_q} := \widehat{\phi}_{\alpha_p,\alpha_q}, \overline{\widehat{\phi}}_{\alpha_p,\alpha_q} := \pi_{E_{\alpha_p}} \circ \overline{\widehat{\phi}}_{\alpha_p,\alpha_q} \circ \pi^{-1}_{E_{\alpha_q}}, \phi_{\alpha_p,\alpha_q} := \pi_{V_{\alpha_p}} \circ \phi_{\alpha_p,\alpha_q} \circ \pi^{-1}_{V_{\alpha_q}}) \ . \tag{4.33}$$

This assignment is well defined since we also required T^n equivariance for $\widehat{\phi}_{\alpha_p,\alpha_q}$ and ϕ_{α_p,α_q}. It is not hard to check that this setting fulfills the properties required in Definition 4.1 and Definition 4.2. Since it is more or less a similar rewriting of what is already be written down we omit this verification here.

To come full circle at the end of this section we aim to state the important Proposition that allows to apply the results of section 4.1 and 4.2 for the moduli space of pseudo-holomorphic curves.

Proposition 4.1

$\mathcal{M}_{l+1}(\beta)$ carries a T^n equivariant Kuranishi structure, is oriented in that sense and the evaluation map at the 0^{th} marked point $ev_0 : \mathcal{M}_{l+1}(\beta) \to L$ is T^n equivariant, weakly continuous and weakly submersive.

Proof: The general way of how equipping $\mathcal{M}_{k+1}(\beta)$ with a K. structure can be found in [FOOO1] section 7.1. The more explicit way, concerning the T^n equivariance, is discussed in [FOOO2] Appendix 2. ■

4.3 Multisections for Kuranishi structures

This section's aim is to introduce the concept of *multisections*. That is defining a set

$$\mathfrak{s} = \{s'_{\alpha_p}\}_{\alpha_p} \tag{4.34}$$

of compatible multisections of the obstruction bundles arising in the context of Kuranishi structures. The $'$ sign shall beware of mixing them up with the sections s_{α_p} of the obstruction bundle, which are can also be seen as (single valued) multisections. Remark that these would be enough, and we thus could skip this section, if $\Gamma_{\alpha_p} = \{e\}$ that is the groups are trivial for all $p \in \mathcal{M}$.
The meaning of multisections is somehow clear, remains to clarify the *compatible* requirement in this context. In order to do so, we start with *l_i-multisections* $s^{l_i}_{\alpha_p}$. Such data consist of an open covering $\bigcup\limits_{i\in I} U_i = V_{\alpha_p}$ of Γ_{α_p} invariant subsets U_i and Γ_{α_p} equivariant maps

$$s^{l_i}_{\alpha_p,i} : U_i \to S^{l_i}(E_{\alpha_p}) := (E^{l_i}_{\alpha_p}/\mathfrak{S}_{l_i}) \times U_i \ . \tag{4.35}$$

Here $\mathfrak{S}_{l_i}$ denotes the symmetric group of order l_i! this means in $S^{l_i}(E_{\alpha_p})$ elements

$$(x_1, ..., x_{l_i}) \sim (x_{\sigma(1)}, ..., x_{\sigma(l_i)}) \tag{4.36}$$

of $\underbrace{E_{\alpha_p} \times ... \times E_{\alpha_p}}_{l_i}$ are identified for $\sigma \in \mathfrak{S}_{l_i}$. From now on we always assume that these l_i-multisections are *liftable* this means that the following diagram commutes

$$\begin{tikzcd} & (E_{\alpha_p} \times U_i)^{l_i} \arrow[d, "\pi^{l_i}_{\alpha_q,i}"] \\ U_i \arrow[ur, "\tilde{s}^{l_i}_{\alpha_p,i}=(\tilde{s}^{l_i}_{\alpha_p,i,1},\dots,\tilde{s}^{l_i}_{\alpha_p,i,l_i})"] \arrow[r, "s^{l_i}_{\alpha_p,i}"'] & S^{l_i}(E_{\alpha_p}) \end{tikzcd}$$

The components $\tilde{s}^{l_i}_{\alpha_p,i,k}$ of the lift $\tilde{s}^{l_i}_{\alpha_p,i}$ are called *branches* of $s^{l_i}_{\alpha_p,i}$.
One defines an equivalence relation for l_i-/m_j-multisections. An thereof arising equivalence class is then simply called a *multisection*

$$s'_{\alpha_p} := [s^{l_i}_{\alpha_p,i}] = [s^{m_j}_{\alpha_p,j}] \tag{4.37}$$

So how is this equivalence relation be defined?
The required representation of Γ_{α_p} on E_{α_p} can be extended to $E^{l_i m_j}_{\alpha_p}$ by embedding

$$\begin{aligned} E^{l}_{\alpha_p} &\longrightarrow E^{l_i m_j}_{\alpha_p} \\ (x_1, ..., x_{l_i}) &\mapsto (\underbrace{x_1, ..., x_1}_{m_j}, ..., \underbrace{x_{l_i}, ..., x_{l_i}}_{m_j}) \ . \end{aligned} \tag{4.38}$$

One therefore naturally gets a Γ_{α_p} equivariant map

$$\iota_{l_i m_j} : S^{l_i}(E_{\alpha_p}) \longrightarrow S^{l_i m_j}(E_{\alpha_p}) \ . \tag{4.39}$$

We say for $s^{l_i}_{\alpha_p,i}, s^{m_j}_{\alpha_p,j}$ to be equivalent ($s^{l_i}_{\alpha_p,i} \sim s^{m_j}_{\alpha_p,j}$) if

$$\iota_{l_i m_j} \circ s^{l_i}_{\alpha_p,i}(x) = \iota_{m_j l_i} \circ s^{m_j}_{\alpha_p,j}(x) \quad \text{for all} \quad x \in U_i \cap U_j, \tag{4.40}$$

that is their images coincide in $S^{l_i m_j}(E_{\alpha_p}) = S^{m_j l_i}(E_{\alpha_p})$.
So far we defined multisections for each $\alpha_p \in I$ and $p \in \mathcal{M}$. The next step is clearly to enlarge them to the whole Kuranishi structure, that is to find how different sections $s'_{\alpha_p}, s'_{\alpha_q}$ $(p \neq q)$ can be considered as compatible. To do so we make use of the isomorphism (4.26). This is possible when assuming that $\mathcal{M}$ has a tangent bundle (in the sense of K. structures). The stated isomorphism helps to slightly modify the exponential map between the neighborhoods

$$B_\epsilon(N_{V_{\alpha_p,\alpha_q}} V_{\alpha_p}) \tag{4.41}$$

(of $\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q}) \subset V_{\alpha_p}$ as the zero section in $N_{V_{\alpha_p,\alpha_q}}V_{\alpha_p}$) and

$$U_\epsilon(\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q})) \tag{4.42}$$

(of $\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q}) \subset V_{\alpha_p}$). Equation (4.26) provides an isomorphism

$$\begin{aligned} TV_{\alpha_p}/T\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q}) \to E_{\alpha_p} \times \phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q})/(\widehat{\phi}_{\alpha_p,\alpha_q}, \phi_{\alpha_p,\alpha_q})(E_{\alpha_q} \times V_{\alpha_p,\alpha_q}) \\ [\widetilde{y}] \mapsto d^{\mathrm{fibre}} s_{\alpha_p}([\widetilde{y}]) \mod \widehat{\phi}_{\alpha_p,\alpha_q}(E_{\alpha_q}) \end{aligned} \tag{4.43}$$

and thus the implicit function justifies the modification of the exponential map. This is done in a way such that one achieves

$$d^{\mathrm{fibre}} s_{\alpha_p}([\widetilde{y}]) \equiv s_{\alpha_p}(y) \mod (\widehat{\phi}_{\alpha_p,\alpha_q}(E_{\alpha_q}))_y \tag{4.44}$$

for

$$[\widetilde{y}] \in TV_{\alpha_p}/T\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q}) \tag{4.45}$$

with

$$\exp(\widetilde{y}) = y \in U_\epsilon(\phi_{\alpha_p,\alpha_q}(V_{\alpha_q})) \ . \tag{4.46}$$

Then choose a representative $(U_i, s^l_{\alpha_q,i})$ of s'_{α_q} with branches $(\widetilde{s}^l_{\alpha_q,i})_k$ and define a new l-multisection $s'_{\alpha_q} \oplus 1$ by $(U_i, (s'_{\alpha_q,i} \oplus 1)^l)$ via

$$\begin{aligned} &(s'_{\alpha_q,i} \oplus 1)^l(y) = \pi^l_{\alpha_q,i} \circ \widetilde{(s'_{\alpha_q,i} \oplus 1)}^{\,l}(y) := \\ =&((\widetilde{s}^l_{\alpha_q,i})_1(\phi^{-1}_{\alpha_p,\alpha_q} \circ \mathrm{pr}(y)) \oplus d^{\mathrm{fibre}} s_{\alpha_p}([\widetilde{y}]), ..., (\widetilde{s}^l_{\alpha_q,i})_l(\phi^{-1}_{\alpha_p,\alpha_q} \circ \mathrm{pr}(y)) \oplus d^{\mathrm{fibre}} s_{\alpha_p}([\widetilde{y}])) \ . \end{aligned} \tag{4.47}$$

Definition 4.7

Multisections s'_{α_p} and s'_{α_q} of $E_{\alpha_p} \times V_{\alpha_p}$ respectively $E_{\alpha_q} \times V_{\alpha_q}$ are said to be compatible if

$$s'_{\alpha_p} \mid_{U_\epsilon(\phi_{\alpha_p,\alpha_q}(V_{\alpha_p,\alpha_q}))} \equiv s'_{\alpha_q} \oplus 1 \ . \tag{4.48}$$

A set $\mathfrak{s} = \{s'_{\alpha_p}\}_{\alpha_p \in I}$ (for all $p \in \mathcal{M}$) of compatible multisections is called a multisection for a space $\mathcal{M}$ with Kuranishi structure.

4.4 Moving forms from source to target spaces

As the headline proposes we want to derive a method of how to transport (pull back and then 'push forward') forms between Lagrangian submanifolds via the moduli space. This section follows the ideas explored by Fukaya et al. in [FOOO2].
Later on we will work with evaluation maps out of our perturbed moduli spaces

$$ev = (ev_1, ..., ev_l, ev_0) : \mathcal{M}^{\mathrm{main}}_{l+1}(L, \beta)^{\mathfrak{s}_\beta} \to L^{l+1}. \tag{4.49}$$

Here the perturbation of $\mathcal{M}^{\mathrm{main}}_{l+1}(L, \beta)$ to $\mathcal{M}^{\mathrm{main}}_{l+1}(L, \beta)^{\mathfrak{s}_\beta}$ (as the zero set of a transversal multisection $\mathfrak{s}_\beta$) is necessary to actually have a smooth manifold structure on it.

See chapter 5.3 for details. Relying on Proposition 4.1 we have our moduli space $\mathcal{M}_{n+1}^{\text{main}}(L,\beta)$ equipped with a T^n equivariant Kuranishi structure and the evaluation map $ev_0 : \mathcal{M}_{l+1}(\beta) \to L$ to be T^n equivariant, strongly continuous and weakly submersive. The meaning of 'source' and 'target' comes into play when we split up (4.49) and define

$$\begin{aligned} ev_s = (ev_1, ..., ev_l) : \mathcal{M}_{l+1}^{\text{main}}(L,\beta)^{\mathfrak{s}_\beta} \to L^l =: L_s \\ ev_t = ev_0 : \mathcal{M}_{l+1}^{\text{main}}(L,\beta)^{\mathfrak{s}_\beta} \to L =: L_t \ . \end{aligned} \tag{4.50}$$

where the s/t stands for 'source' respectively 'target'. In this chapter we still try to stay formal and thus use not closer specified $\mathcal{M}$ for some topological space equipped with T^n equivariant Kuranishi structure

$$((V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p}), (\widehat{\widehat{\phi}}_{\alpha_p,\alpha_q}, \widehat{\phi}_{\alpha_p,\alpha_q}, \phi_{\alpha_p,\alpha_q})) \ . \tag{4.51}$$

The coordinate system shall be a good one and the T^n action is assumed to be free on each coordinate neighborhood. To be able to choose multisections $\mathfrak{s} = \{s'_{\alpha_p}\}_{\alpha_p \in I}$ later we additionally assume $\mathcal{M}$ to have a tangent bundle and to be oriented in the sense of Kuranishi structures. With

$$f_{s/t} : \mathcal{M} \to L_{s/t} \tag{4.52}$$

we denote not closer specified strongly continuous, smooth maps into smooth, compact, oriented manifolds $L_{s/t}$ without boundary. As announced above we additionally want f_t to be weakly submersive. These maps are additionally considered to be T^n equivariant when we have $L_{s/t}$ carrying a T^n action. For L_t we additionally require the action to be free and transitive.
The buildup of this section is twofold. First we try to 'push forward' (by integrating along the fiber) smooth forms ξ_{α_p} of compact support from V_{α_p} to L_t. This 'push forward' will be induced by f_t.
The second part then clarifies the meaning of just using forms

$$\xi_{\alpha_p} \in \Omega_c^k(V_{\alpha_p}) \tag{4.53}$$

that arise as pull backed (induced by f_s) forms τ of the source Lagrangian L_s.
Since the forms ξ_{α_p} depend on the chosen α_p we have to relate them in some way for different $\alpha_p \in I$. We therefore make use of the concept of partition of unity for Kuranishi structures.

First fix an $\alpha_p \in I$. We require T^n to act freely on the K. neighborhoods and therefore can rely to the considerations of section 4.2 that states that $\mathcal{M}/T^n$ carries a naturally induced Kuranishi structure. Choose a multisection

$$^{\mathfrak{t}}\mathfrak{s} = \{^{\mathfrak{t}}s'_{\alpha_p}\}_{\alpha_p \in I} \tag{4.54}$$

of $\mathcal{M}/T^n$ that is *transversal to zero* in the sense of K. structures. This means that all corresponding branches

$$^{\mathfrak{t}}\widetilde{s}^{\,l_i}_{\alpha_p,i,k} : \underbrace{^{\mathfrak{t}}U_i}_{\subset\, V_{\alpha_p}/T^n} \to (E_{\alpha_p}/T^n \times {}^{\mathfrak{t}}U_i)^{l_i} \tag{4.55}$$

are transversal to zero. The lift of it, denoted by $\mathfrak{s} = \{s'_{\alpha_p}\}_{\alpha_p \in I}$, to the initial K. structure can be seen as a T^n equivariant multisection of $\mathcal{M}$ transversal to zero.
Let s'_{α_p} be represented by $(U_i, s^{l_i}_{\alpha_p,i})$. The implicit function theorem thus guarantees that

$$(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0) \subset U_i \tag{4.56}$$

is a smooth manifold for all $k \in \{1, ..., l_i\}$. We want to use the target map f_t for pushing forward forms. The smooth maps

$$f_{t,\alpha_p} \mid_{(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)} \colon (\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0) \to L_t \tag{4.57}$$

shall therefore be submersive. This is true according to the following observation. By Sard´s theorem we find at least one $p_0 \in (\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)$ such that $f_{t,\alpha_p}(p)$ is submersive at the point p_0. Since the T^n action on L_t is assumed to be free and transitive and further that $f_{t,\alpha_p} \mid_{...}$ is T^n equivariant, the property of f to be submersive assigns to all points of $(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)$.
Due to the regular value theorem $f^{-1}_{t,\alpha_{p_0}}(q_0)$ for $q_0 \in L_t$ is a differential manifold of dimension

$$\begin{aligned} \dim\,(f^{-1}_{t,\alpha_{p_0}}(q_0)) &= \dim\,((\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)) - \dim L_t = \\ &= \dim V_{\alpha_p} - \dim E_{\alpha_p} - \dim L_t = \\ &= \text{vir.dim}\,\mathcal{M} - \dim L_t \end{aligned} \tag{4.58}$$

or 0 for $f^{-1}_{t,\alpha_p}(q_0) = \emptyset$. Further we can pick coordinate neighborhoods U_x of $p_0 \in (\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)$, with coordinates $(x_1, ..., x_m)$, and V_x of $q_0 = f_{t,\alpha_p}(p_0) \in L_t$, with coordinates $(x_1, ..., x_n)$ $(n \leq m)$, in a way such that f_{t,α_p} can be written as a projection

$$f_{t,\alpha_p}(x_1, ..., x_n, x_{n+1}, ..., x_m) = (x_1, ..., x_n)\ . \tag{4.59}$$

Further choose $\xi_{\alpha_p} \in \Omega^*_c(V_{\alpha_p})$. The index c suggests that we only use forms whose restriction to each fiber $(f_{t,\alpha_p} \mid_{(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)})^{-1}(q)$ for $q \in L_t$ is of compact support. For a trivialization as in (4.59) $\xi_{\alpha_p} \mid_{(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)}$ can locally be written as a linear combination of elements of

$$\{\underbrace{((f_{t,\alpha_p} \mid_{(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)})^*\theta_x)}_{=:\ \widetilde{\theta}_{\alpha_p,i,k}} f_x(x_1, ..., x_n, x_{n+1}, ..., x_m) dx_{j_1} \wedge ... \wedge dx_{j_r}\} \tag{4.60}$$

for $j_i \in \{n+1, ..., m\}$. Here $\theta_x \in \Omega^l(L_t)$ and $f_x \in C^\infty(U_x, \mathbb{R})$ of compact support for each $(x_1, ..., x_n) \in V_x \subset L_t$ fixed.
We want to map them to forms on L_t. This is done via *integration along the fiber*. Here we only want to recap how this is done. We rely on the ideas of [BoTu] where a detailed description of these methods can be found.
The name integration comes in since we integrate out the fiber components. This means that we reduce the degree of $\xi_{\alpha_p} \mid_{(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)}$ by the dimension of the fiber.

It is locally defined as

$$\begin{aligned} &\Omega_c^*(U_x) \to \Omega^{*-(m-n)}(V_x) \\ &\widetilde{\theta}_{\alpha_p,i,k} f_x(x_1,...,x_n,x_{n+1},...,x_m)dx_{j_1}\wedge ... \wedge dx_{j_r} \mapsto \\ &\begin{cases} \theta_x \int\limits_{\mathbb{R}^{(m-n)}} f_x(x_1,...,x_n,x_{n+1},...,x_m)dx_{n+1}...dx_m & , \text{ for } \quad r=m-n \\ 0 & , \text{ else} \end{cases} . \end{aligned} \tag{4.61}$$

The local definition can be extended to a global degree $-(m-n)$ map

$$(f_{t,\alpha_p})_* : \Omega_c^*((\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)) \to \Omega^*(L_t) \ . \tag{4.62}$$

Assume on the coordinate overlap near $q_0 \in V_x \cap V_y \subset L_t$ one has coordinate functions $\phi_x = (x_1,...,x_n)$ respectively $\phi_y = (y_1,...,y_n)$. For $p_0 \in f^{-1}(q)$ the fiber coordinates $(x_{n+1},...,x_m)$ and $(y_{n+1},...,y_m)$ on $(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)\mid_{V_x}$ respectively $(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)\mid_{V_y}$ give rise to local descriptions for f_{t,α_p}:

$$\begin{gathered} f_{t,\alpha_p}(\Phi_x = (x_1,...,x_n,x_{n+1},...,x_m)) = (x_1,...,x_n) \\ \text{and} \\ f_{t,\alpha_p}(\Phi_y = (y_1,...,y_n,y_{n+1},...,y_m)) = (y_1,...,y_n) \end{gathered} \tag{4.63}$$

We easily check that

$$\begin{aligned} &(\underbrace{\Phi_{xy}}_{=\Phi_x\circ\Phi_y^{-1}})^* \left(\theta_y(y_1,...,y_n) \int\limits_{\mathbb{R}^{(m-n)}} f_y(y_1,...,y_n,y_{n+1},...,y_m)dy_{n+1}...dy_m \right) = \\ &=\theta_y(x_1,...,x_n) \int\limits_{\mathbb{R}^{(m-n)}} f_y(\Phi_{xy}(y_1,...,y_n,y_{n+1},...,y_m))\Phi_{xy}^* dy_{n+1}...\Phi_{xy}^* dy_m = \\ &=\theta_y(x_1,...,x_n) \int\limits_{\mathbb{R}^{(m-n)}} f_y(\Phi_{xy}(y_1,...,y_n,y_{n+1},...,y_m)) \mid \det(\Phi_{xy}) \mid dy_{n+1}...dy_m = \\ &=\theta_x(x_1,...,x_n) \int\limits_{\mathbb{R}^{(m-n)}} f_x(x_1,...,x_n,x_{n+1},...,x_m)dx_{n+1}...dx_m \end{aligned} \tag{4.64}$$

that proves that integration along the fiber is globally defined.

Lemma 4.2

The function $(f_{t,\alpha_p})_*$ (4.62) preserves the property of forms to be closed. Precisely speaking one has

$$(f_{t,\alpha_p})_* \circ d = d \circ (f_{t,\alpha_p})_* \ . \tag{4.65}$$

The induced degree $-(m-n)$ homomorphism

$$((f_{t,\alpha_p})_*)^* : H^*_{cv}((\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)) \to H^*(L_t) \tag{4.66}$$

is actually an isomorphism for $H^*_{cv}(\cdots)$ denoting the cohomology of the complex of forms $\Omega^*_{cv}(\cdot\cdot\cdot)$, that is forms whose restriction to each fiber has compact support.

Proof: [BoTu] ■

The manifold V_{α_p} is covered by the open sets $\{U_i \mid i \in I\}$. So we can use a partition of unity $\{\widetilde{\chi}_i \mid i \in I\}$ subordinated to that covering in order to link the forms $\Omega^k_c((\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0))$ for different i. This yields a possibility to push forward forms

$$\Omega^k_c(V_{\alpha_p}) \to \Omega^{k+\dim L_t - \text{vir.dim}\,\mathcal{M}}(L_t)\ . \tag{4.67}$$

In definition of Kuranishi charts 4.1 we required the group Γ_{α_p} to be finite (wlog. $|\Gamma_{\alpha_p}| = n$). So the replacement

$$\widetilde{\chi}_i(x) \longrightarrow \chi_i(x) := \frac{\chi_i(g_1.x) + ... + \chi_i(g_n.x)}{n} \quad \text{for} \quad g_l \neq g_k \in \Gamma_{\alpha_p} \tag{4.68}$$

provides a Γ_{α_p} invariant partition of unity.

Definition 4.8

The push forward of forms by maps $f_t : V_{\alpha_p} \to L_t$ in the sense of Kuranishi structures is defined as a degree (dim L_t – vir.dim $\mathcal{M}$) map

$$\begin{aligned} (f_t)_* : \Omega^*_c(V_{\alpha_p}) &\longrightarrow \Omega^*(L_t) \\ \xi_{\alpha_p} &\mapsto \frac{1}{\#\Gamma_{\alpha_p}} \sum_{i\in I} \left(\frac{1}{l_i} \sum_{k=1}^{l_i} \underbrace{(f_{t,\alpha_p})_*}_{(4.62)} (\chi_i \cdot \xi_{\alpha_p} \mid_{(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)}) \right) . \end{aligned} \tag{4.69}$$

In the literature (e.g. [FOOO2]) the function is often equivalently denoted as

$$(f_t)_* \equiv ((V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p}), \mathfrak{s}_{\alpha_p}, f_{t,\alpha_p})_*\ . \tag{4.70}$$

Remark 4.2. *For the definition above to be well-defined, the push forward of forms may just depend on the chosen Kuranishi chart, the multisection and the actual target map. Comparably to standard integration theory for manifolds, Definition 4.8 is independent of the representative $(U_i, s^{l_i}_{\alpha_p,i})$ $(i \in I)$ of s'_{α_p} and the corresponding partition of unity $\{\chi_i \mid i \in I\}$:*

Proof: *For another choice* $(V_j, s^{m_j}_{\alpha_p,j})$ *and* $\{\kappa_j \mid j \in J\}$ *one observes*

$$
\begin{aligned}
&\sum_{i\in I}\left(\frac{1}{l_i}\sum_{k=1}^{l_i}(f_{t,\alpha_p})_*(\chi_i\cdot\xi_{\alpha_p}\mid_{(\widetilde{s}^{l_i}_{\alpha_p,i,k})^{-1}(0)})\right)\underbrace{=}_{(4.61)}\\
&=\sum_{i\in I}\left(\frac{1}{l_i}\sum_{k=1}^{l_i}\cdots\int\cdots\chi_i\cdots\right)\underbrace{=}_{\sum_{j\in J}\kappa_j=1}\\
&=\sum_{i\in I}\sum_{j\in J}\left(\frac{1}{l_i}\sum_{k=1}^{l_i}\cdots\int\cdots\chi_i\kappa_j\cdots\right)\underbrace{=}_{\substack{supp\ (\chi_i\kappa_j)\\ compact\ in\ U_i\cap V_j}}\\
&=\sum_{j\in J}\sum_{i\in I}\left(\frac{1}{l_i}\sum_{k=1}^{l_i}\cdots\int\cdots\chi_i\kappa_j\cdots\right)\underbrace{=}_{\sum_{i\in I}\chi_i=1}\sum_{j\in J}\left(\frac{1}{l_i}\sum_{k=1}^{l_i}\underbrace{1}_{=\frac{1}{m_j}\sum_{k'=1}^{m_j}}\cdots\int\cdots\kappa_j\cdots\right)=\\
&=\sum_{j\in J}\left(\underbrace{\frac{1}{l_i}\sum_{k=1}^{l_i}}_{=1}\frac{1}{m_j}\sum_{k'=1}^{m_j}(f_{t,\alpha_p})_*(\kappa_j\cdot\xi_{\alpha_p}\mid_{(\widetilde{s}^{m_j}_{\alpha_p,j,k'})^{-1}(0)})\right)=\\
&=\sum_{j\in J}\left(\frac{1}{m_j}\sum_{k=1}^{m_j}(f_{t,\alpha_p})_*(\kappa_j\cdot\xi_{\alpha_p}\mid_{(\widetilde{s}^{m_j}_{\alpha_p,j,k'})^{-1}(0)})\right)\ .
\end{aligned}
\tag{4.71}
$$

■

Remark 4.3. *In the sense of Lemma 4.2 one has the following relation:*

$$
\begin{aligned}
&\overbrace{d\circ((V_{\alpha_p},E_{\alpha_p},\Gamma_{\alpha_p},\psi_{\alpha_p},s_{\alpha_p}),\mathfrak{s}_{\alpha_p},f_{t,\alpha_p})_*(\xi_{\alpha_p})}^{(I)}=\\
&=\underbrace{(V_{\alpha_p},\ldots)_*(d(\xi_{\alpha_p}))}_{(II)}\pm\underbrace{(\partial V_{\alpha_p},\ldots)_*(\xi_{\alpha_p})}_{(III)}.
\end{aligned}
\tag{4.72}
$$

Proof: Due to presence of the partition of unity in (4.69) we can work locally. According to (4.61) the case $(I)\neq 0$ and $(III)\neq 0$ can not occur for elements of the form

$$\widetilde{\theta}_{\alpha_p,i,k}f_x(x_1,\ldots,x_n,x_{n+1},\ldots,x_m)dx_{j_1}\wedge\ldots\wedge dx_{j_r}\ ;\ j_i\in\{n+1,\ldots,m\}\ . \tag{4.73}$$

Recall that forms of $\Omega^*_{cv}(U_x)$ are build as a linear combination of these.
For forms of the type (4.73) one either has

(i) $r=m-n\Rightarrow(II)=0$ per definition in (4.61) and $(I)=(III)$ due to Lemma 4.2,

or

(ii) $r = m - n - 1 \Rightarrow (I) = 0$ by (4.61) and $(II) = \pm(III)$ since (for $x_{j_i} \in \{x_{n+1}, ..., x_m\}$ pairwise different)

$$\begin{aligned}
&(V_{\alpha_p}, ...)_*(d(\widetilde{\theta}_{\alpha_p,i,k} f_x(x_1, ..., x_n, x_{n+1}, ..., x_m) dx_{j_1} \wedge ... \wedge dx_{j_r})) = \\
=&\underbrace{(d\theta_x) \int\limits_{\mathbb{R}^{(m-n)}} f_x(...) dx_{j_1} ... dx_{j_r}}_{=0} + \\
&+ (-1)^{\deg \theta_x} \theta_x \int\limits_{\mathbb{R}^{(m-n)}} \frac{\partial f_x}{\partial x_{j_{r+1}}}(...) dx_{n+1} ... dx_m \underbrace{=}_{\text{Stokes´ Thm.}} \\
=& \pm (\partial V_{\alpha_p}, ...)_*(\widetilde{\theta}_{\alpha_p,i,k} f_x(x_1, ..., x_n, x_{n+1}, ..., x_m) dx_{j_1} \wedge ... \wedge dx_{j_r}),
\end{aligned} \tag{4.74}$$

or

(iii) $(I) = (II) = (III) = 0$ else for $r \neq \{m - n, m - n - 1\}$. ∎

As stated above we want the $\xi_{\alpha_p} \in \Omega^k_c(V_{\alpha_p})$ to arise as a pulled back form from L_s. The assumed strongly continuous, smooth and T^n equivariant map

$$f_s : \mathcal{M} \to L_s \tag{4.75}$$

is used to define

$$\xi_{\alpha_p} := \chi_{\alpha_p}(f_{\alpha_p,s})^* \tau \tag{4.76}$$

for $\tau \in \Omega^k(L_s)$. The family $\{\chi_{\alpha_p}\}$ represents a partition of unity for Kuranishi structures. One has to rethink the definition of a partition of unity, since we are working over a collection of manifolds V_{α_p} here. The smooth functions of compact support

$$\chi_{\alpha_p} \in C^\infty_c(V_{\alpha_p}, [0, 1]) \quad (\alpha_p \in I) \tag{4.77}$$

are addidionally required to be Γ_{α_p} equivariant. Analogously to standard requirement ($\sum\limits_i \chi_i = 1$) there is an alike way of how the χ_{α_p} sum up to 1. Here the functions for different indices α_p are linked via the coordinate embedding

$$V_{\alpha_p,\alpha_q} \xrightarrow{\phi_{\alpha_p,\alpha_q}} V_{\alpha_p} \tag{4.78}$$

for $\alpha_q \leq \alpha_p$. We do not want to go deeper here and explain how the actual build-up of them is done. The interested reader shall be referred to the literature for example [FOOO2]. Also only stating Lemma 16.6. of the same article simplifies the upcoming constructions in a way that we do not have to worry if such a 'nice' partition actually exits.

Fact 4.1. *For a given T^n equivariant Kuranishi structure one always finds a T^n invariant partition of unity in the sense of Kuranishi structures.*

Proof: [FOOO2] ∎

Remark that ξ_{α_p} in (4.76) is indeed of compact support (necessary for integrating along the fiber) since we required this property for the functions χ_{α_p}. The form ξ_{α_p} just depends on the chosen Kuranishi structure $\{(V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p})\}$ for $\mathcal{M}$, its subordinated partition of unity χ_{α_p} and the source map f_s. We even can get rid of the dependency on the partition of unity when defining the *transport of forms* from L_s to L_t via:

$$\boxed{\begin{aligned} &\Omega^k(L_s) \longrightarrow \Omega^{k+\dim L_t - \text{vir.dim}\,\mathcal{M}}(L_t) \\ (\mathcal{M}, \mathfrak{s}, f_{s,t})_*(\tau) &\equiv (\{(V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p})\}, \mathfrak{s}, f_{s,t})_*(\tau) := \\ &= \sum_{\alpha_p} ((V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p}), \mathfrak{s}_{\alpha_p}, f_{t,\alpha_p})_*(\chi_{\alpha_p}(f_{\alpha_p,s})^*\tau). \end{aligned}} \tag{4.79}$$

The independence of the actual chosen partition $\{\chi_{\alpha_p}\}$ is proven in a similar way as we presented in (4.71). We do not implement this calculation here, since we would need the actual way how the functions χ_{α_p} sum up to 1 for different indices α_p. As above a proof can be found in [FOOO2].

Remark 4.4. *The same argumentation as in Remark 4.3 devolves the transport of forms in (4.79) and one thus gets the following analogon Stokes´ theorem:*

$$\begin{aligned} &d((\mathcal{M}, \mathfrak{s}, f_{s,t})_*(\tau)) = \\ =&(\mathcal{M}, \mathfrak{s}, f_{s,t})_*(d\tau) \pm (\partial\mathcal{M}, \mathfrak{s}, f_{s,t})_*(\tau). \end{aligned} \tag{4.80}$$

Here the Kuranishi structure for $\partial\mathcal{M}$ is naturally induced by the K. structure of $\mathcal{M}$. See Appendix A1. of [FOOO1] for details.

Composition of transport/ Gluing at boundary marked points:

In Proposition 5.4 we see that a union of fiber products of different moduli spaces of the form

$$\begin{aligned} &\mathcal{M}_{k_1+1}(\beta_1)_{ev_0} \times_{ev_i} \mathcal{M}_{k_2+1}(\beta_2) := \\ \{p = (p_1, p_2) \in &\mathcal{M}_{k_1+1}(\beta_1) \times \mathcal{M}_{k_2+1}(\beta_2) \mid ev_0(p_1) = ev_i(p_2)\}. \end{aligned} \tag{4.81}$$

arise as the boundary of

$$\mathcal{M}_{k+1}(\beta) \tag{4.82}$$

for $k_1 + k_2 = k$ and $\beta_1 + \beta_2 = \beta$.
This fact is related to the transport of forms via the question whether construction (4.79) allows to be composed in some way. We assume the following situation (r stands for 'root'; s, t as above for 'source' and 'target'):

$$\begin{tikzcd} L_r & & L_s & & L_t \\ & \mathcal{M}_{rs} \arrow[ul, "ev_r"'] \arrow[ur, "ev_{rs}"] & & \mathcal{M}_{st} \arrow[ul, "ev_{st}"'] \arrow[ur, "ev_t"] & \end{tikzcd}$$

Remark that we replace the general previously used maps f by the more specific notion of the evaluation map ev. If one wants to compose the transportation of forms, we need to require that it already exists for both sides of the diagram above. By assumption the T^n action on L_s and L_t is free and transitive. Further, since ev_{rs} is required to be weakly submersive by assumption, the fiber product

$$\mathcal{M}_{rt} := \mathcal{M}_{rs\ ev_{rs}} \times_{ev_{st}} \mathcal{M}_{st} \tag{4.83}$$

carries a Kuranishi structure. Things get much simplified if we additionally require ev_{st} to be weakly submersive. As described in [FOOO1], we actually would not need this further specification here. Nevertheless we require it here in order to not get too technical.
The Kuranishi neighborhood $(V_{\alpha_p}, E_{\alpha_p}, \Gamma_{\alpha_p}, \psi_{\alpha_p}, s_{\alpha_p})$ for $\mathcal{M}_{rt}$ is defined by:

- $V_{\alpha_{p=(p_1,p_2)}} := \{(x_1, x_2) \in V_{\alpha_{p_1}} \times V_{\alpha_{p_2}} \mid ev_{rs,\alpha_{p_1}}(x_1) = ev_{st,\alpha_{p_2}}(x_2) \in L_s\}$

  (We need the weak submersion property for V_{α_p} to be a manifold.)
- $E_{\alpha_p} := E_{\alpha_{p_1}} \times E_{\alpha_{p_2}}$
- $\Gamma_{\alpha_p} := \Gamma_{\alpha_{p_1}} \times \Gamma_{\alpha_{p_2}}$

  (Recall that Definition 4.3 of strongly continuous maps implies that $ev_{rs,\alpha_{p_1}}$ and $ev_{st,\alpha_{p_2}}$ are $\Gamma_{\alpha_{p_1}}$ respectively $\Gamma_{\alpha_{p_2}}$ invariant.)
- $s_{\alpha_p} := s_{\alpha_{p_1}} \oplus s_{\alpha_{p_2}}$
- $\psi_{\alpha_p} := \psi_{\alpha_{p_1}} \oplus \psi_{\alpha_{p_2}} : s^{-1}_{\alpha_{p_1}}(0)/\Gamma_{\alpha_{p_1}} \times s^{-1}_{\alpha_{p_1}}(0)/\Gamma_{\alpha_{p_1}} \to \mathcal{M}_{rt}$

Similarly a coordinate system is constructed via the direct sum of the given ones. Now one can check (see Appendix A1. of [FOOO1]), that this data defines a Kuranishi structure, with good coordinate system, for $\mathcal{M}_{rt}$. Since all appearing moduli spaces are equipped with Kuranishi structure, we are allowed to state how transports of forms can be composed:

Lemma 4.3 (*Composition of transports*)

The composition of transports of forms give rise to the following commutative diagram

$$\begin{tikzcd} \Omega^{\cdots}(L_r) \arrow[r, "(\mathcal{M}_{rs},\mathfrak{s}_{rs},ev_r,ev_{rs})_*"] \arrow[rr, bend right, "(\mathcal{M}_{rt},\mathfrak{s}_{rt},ev_{rt,r},ev_{rt,t})_*"'] & \Omega^{\cdots}(L_s) \arrow[r, "(\mathcal{M}_{st},\mathfrak{s}_{st},ev_{st},ev_t)_*"] & \Omega^{\cdots}(L_t) \end{tikzcd}$$

.

Proof: [FOOO2] ■

Chapter 5

Lagrangian Floer Cohomology for Torus Fibers

Symplectic manifolds that we are examining in this text are further specified to be compact toric. Due to Delzant this kind can be classified via some polytopial data. Our aim is to explore the behavior of torus fibers arising in this context. For this we assume the reader is already familiar somehow with toric (symplectic) geometry of manifolds.
First we shortly recap some important aspects (moment maps, polytopes...) of toric symplectic geometry and try to illustrate concepts with the help of some examples. For not loosing the relation to our main task, the chapter is tried to be build up in regard to the construction of an A_∞-algebra out of this toric setup. Therefor our focus lies on relevant examples for this context. We consider T^2 actions on 4 dimensional symplectic manifolds namely $S^2 \times S^2$, $\mathbb{C}P^2$ and so on.
The Lagrangian submanifolds on which we aim to attach pseudo-holomorphic curves now arise as level sets of the moment map for interior points of the moment polytope. After more or less just stating necessary facts about the nature of moduli spaces $\mathcal{M}$ we try to incorporate the concepts (perturbation of $\mathcal{M}$ by using multisections etc.) of chapter 4 into this toric setup. It will lead us to define an A_∞-algebra structure by mainly using the idea of transporting forms (4.79) for the definition of the homomorphisms m_k.
Actually we are able to fully calculate (at least for Fano toric manifolds) the described potential function $\mathfrak{PO}$, arising in the context of weak Maurer-Cartan solutions. After these technicalities (especially analyzing the appearing disc and sphere components) we are then able to talk of Lagrangian Floer Cohomology for toric symplectic manifolds. It will directly lead us to chapter 7 where we try to apply this developed theory. By examining derivatives of the already calculated potential function we are able to actually compute the Lagrangian Floer Cohomology and then can face concrete problems ((non-)displaceability questions) of symplectic topology.

5.1 Recollections from Toric Geometry

The section´s purpose lies more on clarifying notations and stating the main theorems necessary for upcoming constructions. For a general introduction to the topic the reader shall be referred to e.g. [McSa] or [Si]. The book [Au] can be seen as the standard textbook about the theory of T^n actions including further supplementary material like Morse theoretic applications. These three references are the main source on which this section is based. More about the algebrao-geometric view on polytopes and toric varieties can be found in [F].

For a given symplectic manifold (M^{2n}, ω) and the (commutative) Lie group $T^m = \underbrace{S^1 \times ... \times S^1}_{m}$ we consider smooth, symplectic actions

$$\begin{aligned} T^m &\to \mathrm{Symp}(M, \omega) \\ g &\mapsto \psi_g \quad . \end{aligned} \tag{5.1}$$

Such actions are further specified to be *hamiltonian* if there is (smooth) moment map

$$\mu : M \to \mathfrak{t}^* = \mathrm{LA}(T^m)^* \cong (R^m)^* \cong \mathbb{R}^m \cong \mathrm{LA}(T^m) = \mathfrak{t} \ . \tag{5.2}$$

For μ the following properties are required:

(i) For $X \in \mathfrak{t}$

$$\begin{aligned} \mu^X : M &\to \mathbb{R} \\ p &\mapsto \langle \mu(p), X \rangle \end{aligned} \tag{5.3}$$

is a hamiltonian function for the *fundamental vector field*

$$\frac{d}{dt} |_{t=0} \ \psi_{\exp(tX)} =: X^{\#} \in \Gamma(TM) \tag{5.4}$$

that is

$$X^{\#} \llcorner \omega = d\mu^X \tag{5.5}$$

(ii)

$$\mu \circ \psi_g = \mu \tag{5.6}$$

Definition 5.1

The data $(M^{2n}, \omega, T^n, \mu)$ describe a symplectic toric manifold if one has:

(i) (M^{2n}, ω) a compact and connected symplectic manifold

(ii) T^n is acting effectively (i.e. $\bigcap_{p \in M} \{g \in T^n \mid \psi_g(p) = p\} = \{e\}$) on M^{2n} (! $2 \cdot \dim T^n = \dim M^{2n}$!)

(iii) μ being the corresponding moment map of the hamiltonian T^n action

We remark that possible moment maps in our case are unique up to a constant $c \in \mathfrak{t}^* \cong \mathbb{R}^n$. For moment maps μ_1, μ_2 we get

$$\mu_1^X(\cdot) - \mu_2^X(\cdot) = c(X, \cdot) : M \to \mathbb{R} \tag{5.7}$$

for a locally constant (due to (5.5)) and since M is assumed to be connected a globally constant function

$$\begin{aligned} c(X, \cdot) : M &\to \mathfrak{t}^* \\ p &\mapsto c(X). \end{aligned} \tag{5.8}$$

This implies $\langle \mu_1(\cdot) - \mu_2(\cdot), X\rangle = c(X)$ meaning

$$\mu_1(\cdot) - \mu_2(\cdot) = c \in \mathfrak{t}^* \tag{5.9}$$

Classification via Delzant polytopes:

Delzant polytopes are described by the intersection of N half-spaces of dimension n in $\mathbb{R}^n$:

$$\Delta^{n,N}_{\lambda_1,\ldots,\lambda_N} \equiv \Delta := \{u \in (\mathbb{R}^n)^* \mid \langle u, v_i\rangle \geq \lambda_i\}. \tag{5.10}$$

Here the vectors

$$v_i \in (\mathbb{Z}^n)_{\text{prim.}} := \{v_i \in \mathbb{Z}^n \mid \nexists\ w \in \mathbb{Z}^n,\ |\underbrace{k}_{\in\mathbb{Z}}| > 1 \text{ with } v_i = kw\} \subset \mathbb{R}^n \tag{5.11}$$

are the *primitive* inward-pointing normal vectors $\in \mathbb{R}^n \cong \mathfrak{t}$.
Delzant polytopes have the properties that each vertex $p \in \Delta$ adjoin to n vertices (*simple*), the edges are of the form (*rational*)

$$p + tu_i; \quad t \in \mathbb{R}_0^+, u_i \in \mathbb{Z}^n \tag{5.12}$$

and the $u_1, ..., u_n$ form an integral basis of $\mathbb{Z}^n$ (*smooth*). Delzant polytopes suit to classify toric symplectic manifolds as follows:

Theorem 5.1

One has the following one-to-one correspondence:

$$\begin{aligned} \{\textit{compact symplectic toric manifolds}\} &\overset{1:1}{\leftrightarrow} \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, T^n, \mu) &\overset{[Ati]}{\to} \mu(M^{2n}) = \Delta^{n,N}_{\lambda_1,\ldots,\lambda_N} \end{aligned} \tag{5.13}$$

with $\mathrm{im}(\mu) = \mathrm{conv}\ \{\mu(x) \mid x \in M,\ \psi_g(x) = x,\ \forall g \in T^n\}$ *and* $\mu^{-1}(u)$ *being connected or empty for* $u \in \Delta$.
Vice versa one has

$$(M_\Delta, \omega_\Delta, T^n, \mu) \overset{[Del]}{\leftarrow} \Delta^{n,N}_{\lambda_1,\ldots,\lambda_N} \tag{5.14}$$

meaning that the corresponding toric manifold $(M_\Delta, \omega_\Delta, T^n, \mu)$ *is unique up to* T^n *equivariant symplectomorphisms.*

We aim to illustrate these concepts on the basis of some examples that will further be important later when we aim to compute Lagrangian Floer Cohomology.

Examples of toric symplectic manifolds and moment polytopes:

a) $\underline{T^2 \hookrightarrow S^2_a \times S^2_b;\text{ for } S^2_r := \{x \in \mathbb{R}^3 \mid \|x\| = r\}}$:

The $T^2 = S^1 \times S^1$ action on $(S^2_a \times S^2_b, \omega = d\theta_1 \wedge dh_1 \oplus d\theta_2 \wedge dh_2)$ is locally given by

$$\begin{aligned}&(e^{i\phi_1}, e^{i\phi_2}).(\sqrt{a^2 - h_1^2} \cdot e^{i\theta_1}, h_1, \sqrt{b^2 - h_2^2} \cdot e^{i\theta_2}, h_2) = \\ =&(\sqrt{a^2 - h_1^2} \cdot e^{i(\phi_1+\theta_1)}, h_1, \sqrt{b^2 - h_2^2} \cdot e^{i(\phi_2+\theta_2)}, h_2)\ .\end{aligned} \tag{5.15}$$

This setup can be seen as a toric symplectic manifold with moment map

$$\mu(\sqrt{a^2 - h_1^2} \cdot e^{i\theta_1}, h_1, \sqrt{b^2 - h_2^2} \cdot e^{i\theta_2}, h_2) = (h_1 + a, h_2 + b) \tag{5.16}$$

since for $i = 1, 2$ one has

$$d\mu^{X_i = e_i} = d(\langle \mu(p), X_i \rangle) = dh_i = \frac{\partial}{\partial \theta_i} \llcorner \omega \overset{\text{loc.}}{=} X_i^{\#} \llcorner \omega \tag{5.17}$$

and

$$\begin{aligned}&\mu(\sqrt{a^2 - h_1^2} \cdot e^{i(\phi_1+\theta_1)}, h_1, \sqrt{b^2 - h_2^2} \cdot e^{i(\phi_2+\theta_2)}, h_2) = \\ =&\mu(\sqrt{a^2 - h_1^2} \cdot e^{i\theta_1}, h_1, \sqrt{b^2 - h_2^2} \cdot e^{i\theta_2}, h_2)\ .\end{aligned} \tag{5.18}$$

The fixed points of this action $(0, -a, 0, -b), (0, a, 0, -b), (0, -a, 0, b), (0, a, 0, b)$ get mapped by μ to $(0, 0), (2a, 0), (0, 2b)$ respectively $(2a, 2b)$.
Therefore the moment polytope (Fig. 5.1), given by their convex hull, is described by

$$\Delta^{2,4}_{\lambda_{1,2}=0, \lambda_3=-2a, \lambda_4=-2b} = \{u \in \mathbb{R}^2 \mid \langle u, e_1 \rangle, \langle u, e_2 \rangle \geq 0; \langle u, -e_1 \rangle \geq -2a; \langle u, -e_2 \rangle \geq -2b\} \tag{5.19}$$

b) $\underline{T^2 \hookrightarrow \mathbb{C}P^2}$:

Similarly to example a) the complex projective space $\mathbb{C}P^2$ can be seen as a toric symplectic manifold. To find the momentum mapping and its image, the Delzant polytope, we first consider the action of T^2 on $\mathbb{C}^3$ given by

$$(e^{i\phi_1}, e^{i\phi_2}).(z_0, z_1, z_2) = (z_0, e^{i\phi_1} z_1, e^{i\phi_2} z_2). \tag{5.20}$$

In this case for $i = 1, 2$ we use the relations

$$d(\mu_i) = d\mu^{X_i} = X_i^{\#} \llcorner \omega \overset{\text{loc.}}{=} \frac{\partial}{\partial \theta_i} \llcorner (\sum_{l=0}^{2} 2r_l d\theta_l \wedge dr_l) = 2r_i dr_i = dr_i^2. \tag{5.21}$$

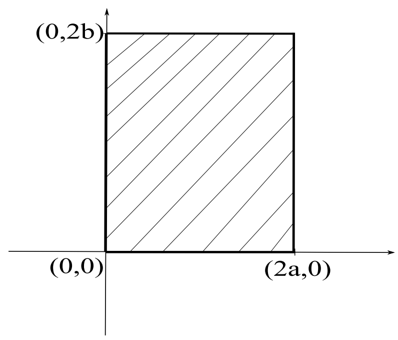


Figure 5.1: moment polytope for $T^2 \hookrightarrow S^2_a \times S^2_b$

So a possible choice for the moment map $\widetilde{\mu}$ would be

$$\begin{aligned} \widetilde{\mu} : \mathbb{C}^3 &\longrightarrow \mathbb{R}^2 \\ (z_0, z_1, z_2) &\mapsto (|z_1|^2, |z_2|^2) \ . \end{aligned} \tag{5.22}$$

In fact the level sets of $\widetilde{\mu}$ are invariant under the torus action so $\widetilde{\mu}$ is a moment map. When considering the sphere $S_{\mathbb{C}} = \{(z_0, z_1, z_2) \mid |z_0|^2 + |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^3$ one has

$$\mathbb{C}P^2 \cong S_{\mathbb{C}}/U(1) \ . \tag{5.23}$$

In this way the hamiltonian T^n action can be assigned to $(\mathbb{C}P^2, \omega_{\mathbb{C}P^2})$ and then is of the form

$$(e^{i\phi_1}, e^{i\phi_2}).[z_0, z_1, z_2] = [z_0, e^{i\phi_1} z_1, e^{i\phi_2} z_2]. \tag{5.24}$$

Remark that $\omega_{\mathbb{C}P^2}$ denotes the Fubini-Study form. The action is hamiltonian with a moment map given by

$$\begin{aligned} \mu : \mathbb{C}P^2 &\longrightarrow \mathbb{R}^2 \\ (z_0, z_1, z_2) &\mapsto \left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right) \end{aligned} \tag{5.25}$$

since $\mu([z_0, z_1, z_2]) = \mu([z_0, e^{i\phi_1} z_1, e^{i\phi_2} z_2])$.
Of course the argumentation here seems to be a bit sloppy. For deducing the moment map for $\mathbb{C}P^2$ in a precise manner one regards

$$\mathbb{C}P^n = S^{2n+1}/S^1 = \mu_0^{-1}(1)/S^1 \tag{5.26}$$

as a reduced space. The form (5.25) is then obtained by symplectic reduction (see e.g. [MarWei]) of

$$(\mathbb{C}^{n+1}, \omega_0, S^1, \mu_0) \tag{5.27}$$

for an action of the form

$$t.(z_0, ..., z_n) = (e^{it} z_0, ..., e^{it} z_n) \tag{5.28}$$

inducing a moment map

$$\mu_0 = |z_0|^2 + ... + |z_n|^2 \tag{5.29}$$

that explains the additional factor in the denominator of (5.25).
The T^n action (5.24) is effective and thus $\mathbb{C}P^2$ can be seen as a toric symplectic manifold. The fixed points of this action are $[z,0,0],[0,z,0],[0,0,z]$ for $z \in \mathbb{C}^*$. According to Theorem 5.1 the moment polytope (Fig. 5.2) is given by the convex hull of

$$\begin{aligned} \mu([z,0,0]) &= (0,0) \\ \mu([0,z,0]) &= (1,0) \\ \mu([0,0,z]) &= (0,1) \end{aligned} \tag{5.30}$$

that is in the sense of Delzant given by

$$\Delta^{2,3}_{\lambda_1=0,\lambda_2=0,\lambda_3=-1} = \{u \in \mathbb{R}^2 \mid \langle u, e_1\rangle, \langle u, e_2\rangle \geq 0; \langle u, -e_1-e_2\rangle \geq -1\} \tag{5.31}$$

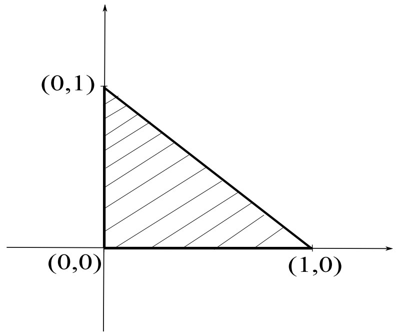


Figure 5.2: moment polytope for $T^2 \hookrightarrow \mathbb{C}P^2$

c) $T^2 \hookrightarrow \overline{\mathbb{C}P^2}_{\mu_v \leq \epsilon}$ (Blow up/ Symplectic cut of $\mathbb{C}P^2$ around fixed points):

Symplectic cutting is a quite general method of constructing new symplectic manifolds out of given ones. Performing this operation around a point (in our case a fixed point of the torus action), can be seen as the analogue of blowing up manifolds in the symplectic category.
Following the ideas of [Le] we are able to understand how symplectic cutting of a toric symplectic manifold $(M^{2n}, \omega, T^n, \mu)$ alters its moment polytope. We shortly recap the stated construction performed around a point $p_0 \in M$. According to Darboux´s theorem we can find an open neighborhood $Op(p_0)$ of p_0 symplectomorphic to $Op(0)$ in $\mathbb{C}^n$ equipped with standard symplectic structure ω_0. This means we can locally think of (M^{2n}, ω) as $(\mathbb{C}^n, \omega_0)$.
The thereof constructible manifold

$$(\mathbb{C}^n \times \mathbb{C}^1, \omega_0 \oplus \frac{i}{2}(dw \wedge d\overline{w})) \tag{5.32}$$

can be equipped with a $T^1 = S^1$ action (with Lie algebra $\mathfrak{s}$) via

$$(z_1, ..., z_n, w) \mapsto (e^{i\theta} z_1, ..., e^{i\theta} z_n, e^{i\theta} w) \ . \tag{5.33}$$

As above in example b) such an action induces a moment map of the form

$$\widetilde{\mu}(z,w) = \mu(z) + |w|^2 = |z_1|^2 + ... + |z_n|^2 + |w|^2 \ . \tag{5.34}$$

Now pick $\epsilon \in \mathbb{R} \cong \mathfrak{s}^*$ being regular value of μ. This property is guaranteed when requiring that S^1 is acting freely on $\mu^{-1}(\epsilon)$. A free S^1 action means that for all $p \in \mu^{-1}(\epsilon)$ their stabilizer $\{g \in S^1 \mid \psi_g(p) = p\}$ is trivial $\{e\}$ and with (5.4) this implies that $d_p\mu$ is surjective and thus p regular. A level set $\{\mu = \epsilon\}$ just consisting of regular points is equivalent to saying that ϵ is a regular value of μ.
Level sets $\{\widetilde{\mu} = \epsilon\} \subset \mathbb{C}^n \times \mathbb{C}^1$ for such an ϵ are given by the disjoint union

$$\{(z,w) \mid \mu(z) < \epsilon, w = e^{i\phi}(\epsilon - \mu(z))^{1/2}\} \sqcup \{(z,w) \mid \mu(z) = \epsilon, w = 0\} \tag{5.35}$$

and are therefore diffeomorphic to

$$\underbrace{\{z \in \mathbb{C}^n \mid \mu(z) < \epsilon\}}_{=:M_{\mu<\epsilon}} \times S^1 \sqcup \mu^{-1}(\epsilon) \ . \tag{5.36}$$

In general a symplectic cut can be performed for arbitrary symplectic manifolds equipped with a hamiltonian S^1 action and moment map μ. It is defined as

$$\overline{M}_{\mu\leq\epsilon} := \{\widetilde{\mu} = \epsilon\}/S^1 = M_{\mu<\epsilon} \sqcup \mu^{-1}(\epsilon)/S^1 \ . \tag{5.37}$$

As already announced above this can be seen as the symplectic blow up at p since we can embedded $M_{\mu<\epsilon}$ in $\overline{M}_{\mu\leq\epsilon}$. The complement of the image is isomorphic to the collapsed boundary $\mu^{-1}(\epsilon)/S^1$.
Due to Lerman [Le] and the fact (5.37) $\overline{M}_{\mu\leq\epsilon}$ carries a canonical symplectic structure ω_ϵ satisfying

$$\omega_\epsilon = \omega \tag{5.38}$$

when restricting it to $M_{\mu<\epsilon} \subset (M, \omega)$.
The requirement above (ϵ regular value respectively S^1 acting freely on $\mu^{-1}(\epsilon)$) was necessary to apply the theorem of Marsden-Weinstein ([MarWei]) that guarantees that $\mu^{-1}(\epsilon)/S^1$ carries the structure of a symplectic manifold.

So let us clarify how this construction fits into the concept of toric symplectic manifolds, namely how the moment polytope of M gets altered when we perform a symplectic cut around a fixed point p_0.
For a given toric symplectic manifold M^{2n} with moment map μ, we consider the S^1 action of $\{\exp(tv)\} \subset T^n$ induced by an inward pointing normal vector $v \in \mathfrak{t}$ of the corresponding Delzant polytope (5.10) of M. According to property (5.5) this action can also be considered to be hamiltonian with induced moment map

$$\mu_v(\cdot) = \langle \mu(\cdot), v \rangle \ . \tag{5.39}$$

The symplectic cut

$$\overline{M}_{\mu\leq\epsilon} = M_{\mu_v<\epsilon} \sqcup \mu_v^{-1}(\epsilon)/S^1 \tag{5.40}$$

can now also be seen as toric symplectic since

$$T^n.M_{\mu_v<\epsilon} = M_{\mu_v<\epsilon} \qquad \text{and} \qquad T^n.\mu_v^{-1}(\epsilon) = \mu_v^{-1}(\epsilon) \ . \tag{5.41}$$

Further by the definition of how S^1 acts we know that this action commutes with the underlying T^n action. The moment polytope for $\overline{M}_{\mu\leq\epsilon}$ is therefore described by

$$\begin{aligned}\Delta^{n,N+1}_{\lambda_1,\dots,\lambda_{N+1}} &= \overbrace{\Delta^{n,N}_{\lambda_1,\dots,\lambda_N}}^{=\mathrm{im}(\mu)} \cap \{u\in(\mathbb{R}^n)^* \mid \langle u,v\rangle \leq \epsilon\} \\ &= \{u\in(\mathbb{R}^n)^* \mid \langle u,v_i\rangle \geq \lambda_i, \langle u,v\rangle \leq \epsilon\}.\end{aligned} \tag{5.42}$$

This means with Delzant´s classification theorem (Theorem 5.1) that we have a correspondence between 'induced' symplectic cuts and parting polytopes by hyperplanes. These considerations yield that for $v = e_2$ the moment polytope (Fig. 5.3) of

$$T^2 \hookrightarrow \overline{\mathbb{C}P^2}_{\mu_{v=e_2}\leq\epsilon} \tag{5.43}$$

around the fix point $p_0 = [0,0,z]$ $(z\in\mathbb{C}^*)$ is described by

$$\Delta^{2,4}_{\lambda_{1,2}=0,\lambda_3=-1,\lambda_4=-\epsilon} = \{u\in\mathbb{R}^2 \mid \langle u,e_i\rangle \geq 0; \langle u,-e_1-e_2\rangle \geq -1; \langle u,-e_2\rangle \geq -\epsilon\}\ . \tag{5.44}$$

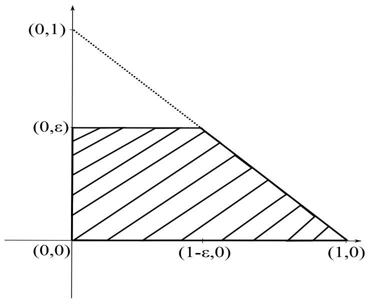


Figure 5.3: moment polytope for $T^2 \hookrightarrow \overline{\mathbb{C}P^2}_{\mu_v\leq\epsilon}$

d) $\underline{T^2 \hookrightarrow \text{Hirzebruch surface } (H_k, \omega_0|_{H_k}) \subset (\mathbb{C}P^1\times\mathbb{C}P^2, \omega_0 = \omega_{\mathbb{C}P^1}\oplus\omega_{\mathbb{C}P^2})}$:

Extending the ideas of example (b) we first try to find the moment map for the effective T^2 action

$$(e^{i\phi_1}, e^{i\phi_2}).([a,b],[x,y,z]) = ([e^{i\phi_1}a,b],[e^{i\cdot k\cdot\phi_1}x,y,e^{i\phi_2}z]) \tag{5.45}$$

for $k\in\mathbb{N}_0$ fixed. When considering the torus action separately for $\mathbb{C}P^1$ and $\mathbb{C}P^2$ we get the moment maps

$$\mu_1([a,b]) = \left(\frac{|a|^2}{|a|^2+|b|^2}, 0\right) \tag{5.46}$$

respectively

$$\mu_2([x,y,z]) = \left(k\cdot\frac{|x|^2}{|x|^2+|y|^2+|z|^2}, \frac{|z|^2}{|x|^2+|y|^2+|z|^2}\right)\ . \tag{5.47}$$

The first case is done similarly like in example b) and for the action on $\mathbb{C}P^2$ we remark that now (5.21) is of the form

$$d\mu_2^{X_x} = X_x^{\#} \llcorner \omega \overset{\text{loc.}}{=} (k \cdot \frac{\partial}{\partial \theta_x}) \llcorner \omega = k \cdot dr_x^2 \tag{5.48}$$

which justifies the factor k in the first component of (5.47).
For the diagonal action $T^2 \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}P^2$ given by (5.45) we get the moment map

$$\mu([a,b],[x,y,z]) = \mu_1 + \mu_2 = \left(\frac{|a|^2}{|a|^2+|b|^2} + k \cdot \frac{|x|^2}{|x|^2+|y|^2+|z|^2}, \frac{|z|^2}{|x|^2+|y|^2+|z|^2} \right) \tag{5.49}$$

which is clearly invariant under the T^2 action.
Here we can simply take the sum of the individual moment maps μ_1, μ_2 since ω_{H_k} is declared by a (restriction of a) direct sum.
Now consider the subset

$$H_k := \{([a,b],[x,y,z]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \mid F([a,b],[x,y,z]) := a^k y - b^k x = 0\} \ . \tag{5.50}$$

Remark that F is globally defined since

$$F([\lambda a, \lambda b],[\kappa x, \kappa y, \kappa z]) = F([a,b],[x,y,z]) \quad \text{for} \quad \lambda, \kappa \in \mathbb{C} \ . \tag{5.51}$$

The inverse function theorem states that zero level set of the homogeneous polynomial F is a complex submanifold of dimension $\dim_{\mathbb{C}} = 2$. It is called the *Hirzebruch surface* H_k. As $\mathbb{C}P^1 \times \mathbb{C}P^2$ is Kähler H_k is in particular a symplectic manifold.
By the definition of the torus action

$$(e^{i\phi_1} a)^k y - b^k (e^{ik \cdot \phi_1} x) = e^{ik \cdot \phi_1}(a^k y - b^k x) \tag{5.52}$$

we have $F(T^n.H_k) = 0$. This invariance of the level sets $\{F = 0\}$ allows to restrict the above considered action to H_k with moment map $\mu = (5.49)$. The setup can thus be seen as a toric symplectic manifold $(H_k, \omega_0|_{H_k}, T^2, \mu)$. The convex hull of the image of the fixed points (remark $([0,z],[w,0,0]), ([z,0],[0,w,0]) \notin H_k$)

$$\begin{aligned} ([0,z],[0,0,w]) &\xrightarrow{\mu} (0,1) \\ ([0,z],[0,w,0]) &\xrightarrow{\mu} (0,0) \\ ([z,0],[0,0,w]) &\xrightarrow{\mu} (1,1) \\ ([z,0],[w,0,0]) &\xrightarrow{\mu} (k+1,0) \end{aligned} \tag{5.53}$$

gives rise to a moment polytope (Fig. 5.4) of the form

$$\begin{aligned} \Delta^{2,4}_{\lambda_{1,2}=0, \lambda_3=-1, \lambda_4=-(1+1/k)} = \{u \in \mathbb{R}^2 \mid \langle u, e_1 \rangle, \langle u, e_2 \rangle \geq 0; \langle u, -e_2 \rangle \geq -1; \\ \langle u, -e_1 - ke_2 \rangle \geq -(k+1)\} \ . \end{aligned} \tag{5.54}$$

e) $\underline{T^2 \hookrightarrow \text{deg. Hirzebruch surface } (H_k(\alpha), \omega_{k,\alpha}|_{H_k(\alpha)} = (\omega^{k\alpha}_{\mathbb{C}P^1} \oplus \omega^{(1-\alpha)}_{\mathbb{C}P^2})|_{H_k(\alpha)})}$:

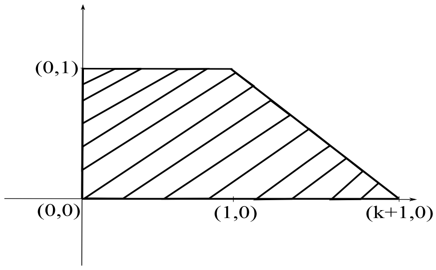


Figure 5.4: moment polytope for $T^2 \hookrightarrow H_k$

We aim to get a bit more general perspective on the discussion above. That is observing the case for a family of symplectic structures

$$\{\omega_{\mathbb{C}P^1}^{k\alpha} := k\alpha \cdot \omega_{\mathbb{C}P^2}\}_{k\in\mathbb{N},\alpha\in(0,1)} \ ; \tag{5.55}$$

$$\{\omega_{\mathbb{C}P^2}^{(1-\alpha)} := (1-\alpha) \cdot \omega_{\mathbb{C}P^2}\}_{\alpha\in(0,1)} \tag{5.56}$$

on $\mathbb{C}P^1 \times \mathbb{C}P^2$.
Remark that multiplication with a constant $\in \mathbb{R}$ does not destroy the property of being a symplectic form. In the following we denote this specific toric manifold (arising like in example d) as the complex submanifold $F^{-1}(0)$ and with T^2 acting as in (5.45)) by $H_k(\alpha)$.
For moment maps the underlying symplectic structure is involved via

$$d\mu^X = X^\# \llcorner \omega, \tag{5.57}$$

so μ of example d) gets generalized to

$$\mu^{k\alpha}([a,b],[x,y,z]) = \Bigg(k \cdot \alpha \cdot \frac{|a|^2}{|a|^2+|b|^2} + (1-\alpha) \cdot k \cdot \frac{|x|^2}{|x|^2+|y|^2+|z|^2}, \\ (1-\alpha) \cdot \frac{|z|^2}{|x|^2+|y|^2+|z|^2} \Bigg) \ . \tag{5.58}$$

Again considering the image of the fixed points

$$\begin{aligned} ([0,z],[0,0,w]) &\overset{\mu^{k\alpha}}{\to} (0,(1-\alpha)) \\ ([0,z],[0,w,0]) &\overset{\mu^{k\alpha}}{\to} (0,0) \\ ([z,0],[0,0,w]) &\overset{\mu^{k\alpha}}{\to} (k\alpha,(1-\alpha)) \\ ([z,0],[w,0,0]) &\overset{\mu^{k\alpha}}{\to} (k,0) \end{aligned} \tag{5.59}$$

that span a polytope (Fig. 5.5) described by

$$\Delta^{2,4}_{\lambda_{1,2}=0,\lambda_3=\alpha-1,\lambda_4=-k} = \{(u_1,u_2) \in \mathbb{R}^2 \mid u_i \geq 0; u_2 \leq 1-\alpha; u_1 + ku_2 \leq k\} \ . \tag{5.60}$$

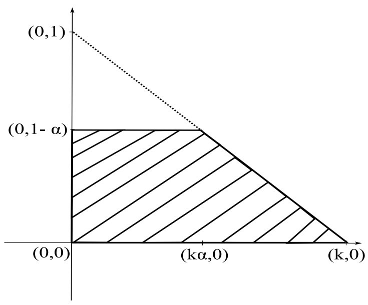


Figure 5.5: moment polytope for $H_k(\alpha)$

Conclusion of the examples:

When comparing the moment polytopes of the sphere product $S^2_a \times S^2_b$ (5.19) for $a = b = 1/2$ and of the Hirzebruch surface H_k (5.54) for $k = 0$ one sees that they coincide. Remark that in the literature one normally gets $a = b = 1$ when scaling both symplectic forms consistently. For later purposes (when calculating the potential function in section 6.2.2) the way we describe things here is more helpful and so we just leave results as they are written in a) and d).
The same holds for the polytopes of the symplectic cut (one point blow up around $[0, 0, 1]$) $\overline{\mathbb{C}P^2}_{\mu_{v=e_2}\leq\epsilon}$ (5.44) and of the degenerated Hirzebruch surface $H_k(\alpha)$ (5.60) for $1 - \alpha = \epsilon$ and $k = 1$.
According to Theorem 5.1 we deduce that these respective manifolds are symplectomorphic meaning

$$\begin{aligned} S^2_{1/2} \times S^2_{1/2} &\sim_D H_0 \\ \overline{\mathbb{C}P^2}_{\mu_{v=e_2}\leq 1-\alpha} &\sim_D H_1(\alpha) \end{aligned} \tag{5.61}$$

that is they are equivalent in the sense of Delzant.

Lagrangian Torus Fibers arising as preimages of interior points:

In order to detect torus fibers of the momentum fibration $\mu : M^{2n} \to \Delta^n$ we first clarify that preimages $\mu^{-1}(p)$ of regular values $p \in \Delta$ are indeed submanifolds of M^n.
It can be proven (see e.g. [Au]) that the dimension of the facet of Δ to which p belongs coincide with the rank of

$$d_x\mu : T_xM \to \mathfrak{t}^* \quad \text{for all} \quad x \in \mu^{-1}(p) \ . \tag{5.62}$$

This means in particular that μ is a submersion for all x with $\mu(x) \in \mathring{\Delta}$. Therefore $\mu^{-1}(p)$ is a compact (M is assumed to be compact), connected (Theorem 5.1) manifold of dimension $2n - n = n$ for all $p \in \mathring{\Delta}$.

We want these fibers to be a torus. In order to apply the Arnold-Liouville theorem (Theorem 5.2) we have to check that $C^\infty(M, \mathbb{R})$ maps of the form

$$\mu^X = \langle \mu(\cdot), X \rangle \tag{5.63}$$

commute with respect to the Poisson bracket, that is

$$\{\mu^X, \mu^Y\} \equiv 0 \ . \tag{5.64}$$

We remark that for $X, Y \in \mathfrak{t}$ and $[.,.]$ denoting the Lie bracket of $\mathfrak{t}$ respectively $\Gamma(TM)$ the identity

$$[X, Y]^\# = [X^\#, Y^\#] \tag{5.65}$$

holds per definition of $X^\#$, being the vector field generated by $\psi_{\exp(tX)}$.
Recall the following identity

$$d(\mu^{[X,Y]}) = [X, Y]^\# \llcorner \omega = [X^\#, Y^\#] \llcorner \omega \underbrace{=}_{(*)} d(\{\mu^X, \mu^Y\}) \tag{5.66}$$

The last equality $(*)$ holds when we regard M as a Poisson manifold via

$$\{f, g\} := \omega(X_f, Y_g) \ . \tag{5.67}$$

for $f, g \in C^\infty(M, \mathbb{R})$ and $X_f, Y_g \in \Gamma(TM)$ their corresponding hamiltonian vector field. For moment maps this definition is thus written as

$$\{\mu^X, \mu^Y\} := \omega(X^\#, Y^\#) \ . \tag{5.68}$$

Then for the vector field $[X^\#, Y^\#]$ we have

$$\begin{aligned} [X^\#, Y^\#] \llcorner \omega &= (\frac{d}{dt} \mid_{t=0} (\phi^t_{X^\#})^* Y^\#) \llcorner \omega = \\ &= \frac{d}{dt} \mid_{t=0} d(\mu^Y \circ \phi^{-t}_{X^\#}) = \\ &= -d(d\mu^Y(X^\#)) = -d(\omega(Y^\#, X^\#)) = \\ &= d\{\mu^X, \mu^Y\} \end{aligned} \tag{5.69}$$

and therefore $[X^\#, Y^\#]$ is hamiltonian for the function $\{\mu^X, \mu^Y\}$. This proves equality $(*)$ of (5.66).
Since T^n is commutative we have $[X, Y] = 0$ for all X, Y and so $\{\mu^X, \mu^Y\}$ is a locally and since we assume M to be connected a globally constant function $C^\infty(M, \mathbb{R})$. According to the considerations above about the rank of $d_x\mu$ we know that μ^X has at least one critical point and thus

$$\{\mu^X, \mu^Y\} \equiv 0 \ . \tag{5.70}$$

Theorem 5.2 (*Arnold-Liouville*)

Assume a smooth function

$$f : M \to \mathbb{R}^n \tag{5.71}$$

on a symplectic manifold (M^{2n}, ω) with commuting component functions f_i

$$\{f_i, f_j\} = 0 \tag{5.72}$$

is given. Then compact connected manifolds of dimension n that arise as level *sets*

$$\{f = const.\} \tag{5.73}$$

are diffeomorphic to T^n.

Proof: [A] ■

The above theorem can be applied in our case. For the standard basis (X_i) of $\mathbb{R}^n \cong \mathfrak{t}$ we have $\mu = (\mu_1 = \mu^{X_1}, ..., \mu_n = \mu^{X_n})$ with

$$\{\mu_i, \mu_j\} = 0 \; . \tag{5.74}$$

Therefore $\mu^{-1}(p)$ is diffeomorphic to T^n (and therefore orientable) for all $p \in \mathring{\Delta}$.

We finally check that the torus fibers over interior points of Δ are indeed Lagrangian tori. For all x in the level sets $\mu^{-1}(p)$ one has

$$\begin{aligned} \ker(\omega \mid_{\mu^{-1}(p)})_x &= T_x\mu^{-1}(p) \cap (T_x\mu^{-1}(p))^\omega = \\ &= \ker\, d_x\mu \cap (\ker\, d_x\mu)^\omega \end{aligned} \tag{5.75}$$

for the tangent map $d\mu : TM \to \mathfrak{t}^*$. The stated assertion then follows by counting dimensions and the fact that $d_x\mu$ is surjective. Precisely speaking we further have

$$\ker\, d_x\mu = (T_x(T^n.x))^\omega \tag{5.76}$$

for

$$T^n.x := \{\psi_g(x) \mid g \in T^n\} \subset M \tag{5.77}$$

denoting the orbit through x. This is true since the vanishing of $d_x\mu(X)$ corresponds to $\langle d_x\mu(X), Y\rangle = 0$ for all $Y \in \mathfrak{t}$. Due to (5.5) this holds if and only if

$$Y_x^\# \llcorner \omega_x(X) = 0 \; . \tag{5.78}$$

Since $T_x(T^n.x)$ is generated by n linear independent vectors $Y_x^\#$ of the fundamental vector fields, assertion (5.76) follows.
We get

$$(5.75) = \ker\, d_x\mu \cap T_x(T^n.x) = \ker\, d_x\mu = T_x\mu^{-1}(p) \; . \tag{5.79}$$

Here the second equality holds since due to the required invariance (5.6) of μ we have

$$\mu(T^n.x) = p \quad \text{for all} \quad x \in \mu^{-1}(p) \tag{5.80}$$

and thus $d_x\mu(T_x(T^n.x)) = 0$. In summary we have

$$\boxed{\begin{array}{c} \mu^{-1}(p) \text{ is a compact, orientable Lagrangian torus} \\ \text{inside the toric manifold } (M^{2n}, \omega) \\ \text{for all } p \in \mathring{\Delta} \end{array}} \tag{5.81}$$

5.2 Examination/Perturbation of Moduli Spaces

After a detailed discussion about construction and examination of the 'target' (toric symplectic manifold and Lagrangian subtori $L(p)$), we now try to get a better insight into the behavior of the holomorphic curves that get mapped onto it. We affiliate the discussion of section 2.3 where we clarified which objects are actually contained in the corresponding moduli space

$$\mathcal{M}_{l+1}^{(\mathrm{main}),(\mathrm{reg})}(L(p),\beta) \equiv \mathcal{M} \tag{5.82}$$

for $\beta \in \pi_2(M, L(p)) \overset{\text{Hurewicz}}{\longrightarrow} H_2(M, L(p); \mathbb{Z})$.
We aim to get a better insight into the properties of these moduli spaces. By picking up the ideas of chapter 4 this is done in order to define the stated algebraic concepts (A_∞-algebra, potential function $\mathfrak{PO}$, Maurer-Cartan solutions, etc.) of chapter 3 out of $\mathcal{M}$. The proofs of the facts that we state here are more or less the content of the second part (i.e. chapter 7 and 8) of [FOOO1]. Due to its length and the amount of technicality, a precise description about these ideas would go beyond the scope of this text. Its relevance lies more on clarifying the language of Lagrangian Floer Cohomology and especially how to find applications for facing current problems in symplectic topology. Nevertheless for the interested reader we indicate where the relevant proofs can be found in the literature.
Due to the construction of chapter 7.1 of [FOOO1] we have:

Proposition 5.1

For $l \neq 0$ the moduli space $\mathcal{M}_l^{main}(L(p),\beta)$ can be equipped with an oriented Kuranishi structure. Further it carries a topology due to which it is compact (after compactification) and Hausdorff.

To stay nearby our toric setup we remember proposition 4.1, namely that the Kuranishi structure and especially the evaluation map can be chosen to be T^n equivariant. We additionally assume $L \subset M$ to be *relatively spin* meaning that there exists a class $\alpha \in H^2(M; \mathbb{Z}_2)$ such that for its restriction to L we have

$$\alpha \mid_L = w_2(L). \tag{5.83}$$

Here $w_i(L)$ denotes the i-th Stiefel-Whitney class of TL. In such a case we further adopt the results of chapter 8 of [FOOO1] and get:

Proposition 5.2

$\mathcal{M}_l^{main}(L(p),\beta)$ carries an orientation canonically induced by a chosen relative spin structure for $L \subset M$.

Remark that in the toric setup $L(p)$ is always relatively spin. It is diffeomorphic to

$$T^n = S^1 \times ... \times S^1 \tag{5.84}$$

which in turn means that it is parallelizable ($TT^n \cong T^n \times \mathbb{R}^n$) and we therefore get

$$w_2(TL(p)) = w_2(TT^n) = 0 \ . \tag{5.85}$$

Since we are working in the prescribed toric setup further helpful properties can be deduced for this specified situation.
Consider a finite set of homology classes

$$\{\beta_1, ..., \beta_N\} \tag{5.86}$$

for $\beta_i \in H_2(M, L(p); \mathbb{Z})$ defined by the intersection product

$$\beta_i \bullet \underbrace{[\mu^{-1}(\partial\Delta_j)]}_{\in H_{2n-2}(M,L(p);\mathbb{Z})} = \delta_{i,j}. \tag{5.87}$$

Here μ is the moment map and

$$\partial\Delta_i := \{p \in \partial\Delta \mid \langle p, v_i\rangle - \lambda_i = l_i(u) = 0\} \tag{5.88}$$

describes the i-th of the N faces of Δ.
As we will see in chapter 5.4 holomorphic discs of class β_i are well understood, meaning that due to the work of C.-H. Cho and Y.-G. Oh in [CO] we know their Maslov index ($\mu(\beta_i) = 2$) and their symplectic volume ($\omega(\beta_i) = 2\pi l_i(u)$).
For our purpose we rely to Theorem 11.1. of [FOOO2] where these holomorphic disks appear as follows and therefore are quite useful to understand moduli spaces of Maslov index 2 type holomophic maps:

Proposition 5.3

(i) $\mu(\beta) < 0$ *or* $\mu(\beta \neq 0) = 0$ *implies* $\mathcal{M}^{main,reg}_{l+1}(L(p), \beta) = \emptyset$

(ii) $\mu(\beta \neq \beta_i) = 2$ *implies* $\mathcal{M}^{main,reg}_{l+1}(L(p), \beta) = \emptyset$

(iii) $\mathcal{M}^{main,reg}_{1}(L(p), \beta_i) = \mathcal{M}^{main}_{1}(L(p), \beta_i)$ *and* $\mathcal{M}^{main,reg}_{1}(L(p), \beta)$ *are Fredholm regular*

(iv) $ev : \mathcal{M}^{main}_{1}(L(p), \beta_i) \to L(p)$ *is an orientation preserving diffeomorphism*

(v) $ev : \mathcal{M}^{main,reg}_{1}(L(p), \beta) \to L(p)$ *is a submersion*

(vi) $ev : \mathcal{M}^{main}_{1}(L(p), \beta) \neq \emptyset$ *implies that* β *can be decomposed into disc-* β_i *and sphere components* $\alpha_j \in H_2(M; \mathbb{Z})$ *i.e.*

$$\beta = \sum_{i=1}^{N} k_i\beta_i + \sum_j \alpha_j \tag{5.89}$$

such that $\exists\ i_0$ *with* $k_{i_0} \neq 0$

Proof: [FOOO2] ■

As already described in section 4.4 for deriving a possibility of transporting forms from sources to targets we need to perturb $\mathcal{M}$ by using multisections $\mathfrak{s}$. This provides

$$ev : \mathcal{M}_1(L(p), \beta)^{\mathfrak{s}_\beta} \to L(p) \tag{5.90}$$

to be a submersion. Adopting the ideas of chapter 7.2 of [FOOO1] yields the following result

Proposition 5.4

It is possible to choose multisections $\mathfrak{s}_{\beta,l+1} \equiv \mathfrak{s}$ for $\mathcal{M}^{main}_{l+1}(L(p),\beta)$ such that

1. $\mathfrak{s}_{\beta,l+1} \pitchfork 0$ *(transversal) and* $\mathfrak{s}_{\beta,l+1}(T^n.\cdot) = \mathfrak{s}_{\beta,l+1}(\cdot)$ *(T^n invariant)* .

Additionally the restriction $\mathfrak{s}_{\beta,l+1}\mid_{\partial\mathcal{M}^{main}_{l+1}(L(p),\beta)}$ equals the restricted fiber product $\mathfrak{s}_{\beta_1,l_1+1} \times \mathfrak{s}_{\beta_2,l_2+1}$. This equality holds on $\partial\mathcal{M}^{main}_{l+1}(L(p),\beta)$ that is given by the union of fiber products

$$\bigcup_{l_1+l_2=l+1}\ \bigcup_{\beta_1+\beta_2=\beta}\ \bigcup_{j=1}^{l_2} \mathcal{M}^{main}_{l_1+1}(L(p),\beta_1)_{ev_0} \times_{ev_j} \mathcal{M}^{main}_{l_2+1}(L(p),\beta_2) \tag{5.91}$$

Proof: [FOOO2] ∎

For proving the assertion about the equality of the respective multisections one first deduces it for $\mathfrak{s}_{\beta,1}$ of $\partial\mathcal{M}^{\text{main}}_1(L(p),\beta)$ and then makes use of the forgetful map:

$$\begin{aligned} \mathfrak{forget}_0 : \mathcal{M}^{\text{main}}_{l+1}(L(p),\beta) &\to \mathcal{M}^{\text{main}}_1(L(p),\beta) \\ (w; z_0, ..., z_k) &\mapsto (w; z_0) \end{aligned} \tag{5.92}$$

This progression can be followed since one is able to show (Lemma 11.2. of [FOOO2])

$$\mathfrak{s}_{\beta,l+1} = \mathfrak{forget}_0^*(\mathfrak{s}_{\beta,1}). \tag{5.93}$$

We further remark that the Proposition above is formulated a bit sloppy. The correct way would be to regard only β with $\omega(\beta) < E$ for fixed energies E. It will yield us an $A_{n(E),K(E)}$-algebra structure with

$$\lim_{E\to\infty}(n(E),K(E)) = (\infty,\infty). \tag{5.94}$$

Tough one is able to show that the therein arising homomorphisms $m_k^{(E)}$ can be extended to a A_∞ structure. Since as in our case the most interesting ingredient namely the form of the potential function $\mathfrak{PO}$ does not depend on the chosen E we neglect these technical facts and take proposition 5.4 as it is stated above.

5.3 Construction of an A_∞-algebra

As described in chapter 5.2 compact, orientable Lagrangian tori

$$L(p) := \mu^{-1}(p) \quad (p \in \mathring{\Delta}) \tag{5.95}$$

arise as T^n orbits in the corresponding toric symplectic manifold (M^{2n},ω,T^n,μ). For such a setup and the usage moduli spaces of stable maps from Riemann surfaces with boundary attaching these Lagrangians we aim to define an A_∞-algebra

structure. Again this chapter relies on the ideas presented in [FOOO1], [FOOO2] and [FOOO4].
In the following for the unfiltered R module we take the de Rham cohomology group

$$H^*_{dR}(L(p), \mathbb{R}) \tag{5.96}$$

as a free graded module over $\mathbb{R}$. The subscript 'de Rham' is mostly omitted in this text except we are considering different types of cohomology at the same time.
As a first step we declare homomorphisms

$$\overline{m}_l \equiv m_{l,\beta} : \bigoplus_{r_1,\dots,r_l} H^{r_1}(L(p), \mathbb{R}) \otimes \dots \otimes H^{r_l}(L(p), \mathbb{R}) \to H^*(L(p), \mathbb{R}) \tag{5.97}$$

of degree $\sum_{i=1}^{l} r_i + 1 - \mu(\beta)$ for $\beta \in \pi_2(M, L(p))$.
Remark that since $L(p)$ is a T^n orbit, when choosing a T^n equivariant metric on $L(p)$, we can always find a differential form

$$\alpha \in \Omega^r(L(p)) \tag{5.98}$$

that is harmonic ($\Delta\alpha = 0$) if and only if it is T^n equivariant that is

$$\alpha(\psi_g(x)) = (\psi_{g^{-1}})^* \alpha(x) \quad \text{for} \quad g \in T^n. \tag{5.99}$$

According to the work of W. V. D. Hodge (e.g. [La]) we have an isomorphism

$$H^r(L(p), \mathbb{R}) \cong \Omega^r_{\text{harmonic}}(L(p)) \ . \tag{5.100}$$

So in the following for an equivalence class $[\alpha_i] \in H^{r_i}(L(p), \mathbb{R})$ we always take its unique harmonic and T^n equivariant representative

$$\alpha_i \in \Omega^{r_i}(L(p)) \ . \tag{5.101}$$

We will mostly neglect the homology class brackets $[\cdot]$ and write α likewise for harmonic forms or homology classes if no confusion can occur.
To adopt the results of chapter 4.4 and 5.2 we denote

$$\begin{aligned} \mathcal{M} =& \mathcal{M}^{\text{main}}_{l+1}(L(p), \beta)^{\mathfrak{s}_\beta} \\ L_{\text{source}} :=& \underbrace{L(p) \times \dots \times L(p)}_{l} \\ ev_{\text{source}} :=& (ev_1, \dots, ev_l) \\ L_{\text{target}} :=& L(p) \\ ev_{\text{target}} :=& ev_0 \end{aligned} \tag{5.102}$$

for evaluation maps

$$\begin{aligned} ev_i : \mathcal{M} &\to L^n \subset M^{2n} \\ [(w; p_0, \dots, p_l)] &\mapsto w(p_i). \end{aligned} \tag{5.103}$$

Further recall Proposition 4.1, namely that the evaluation map at the 0^{th} marked point $ev_0 : \mathcal{M}_{l+1}(\beta) \to L$ is T^n equivariant, weakly continuous and weakly submersive.
With the transport of forms (4.79) we are to able define degree $1 - \mu(\beta)$ maps

$$m_{l,\beta} : (C[1])^{r_1} \times ... \times (C[1])^{r_l} \to (C[1])^{\sum_{i=1}^{l} r_i - \mu(\beta)+1} \tag{5.104}$$

by setting $C^{r_1} := H^{r_1}(L(p), \mathbb{R})$ and declaring

$$\begin{aligned} m_{l,\beta} : H^{r_1+1}(L(p), \mathbb{R}) \otimes ... \otimes H^{r_l+1}(L(p), \mathbb{R}) \to H^{\sum_{i=1}^{l}(r_i+1)+\dim L_t - \text{vir.dim}\mathcal{M}=(*)}(L(p), \mathbb{R}) \\ (\alpha_1, ..., \alpha_l) \mapsto (\mathcal{M}, \mathfrak{s}_\beta, ev_{s,t})_*(\alpha_1 \times ... \times \alpha_l) \end{aligned} \tag{5.105}$$

which indeed is of degree

$$(*) = \sum_{i=1}^{l}(r_i + 1) + n - (n + \mu(\beta) + l + 1 - 3) = \sum_{i=1}^{l} r_i - \mu(\beta) + 2 \tag{5.106}$$

that is

$$\text{im}(m_{l,\beta}) \in H^{\sum_{i=1}^{l} r_i - \mu(\beta)+2}(L(p), \mathbb{R}) = (C[1])^{\sum_{i=1}^{l} r_i - \mu(\beta)+1} . \tag{5.107}$$

Remark that the image of $m_{l,\beta}$ is a T^n equivariant form and therefore again harmonic. This holds since we are able to choose the Kuranishi structure (Prop. 4.1) and $\mathfrak{s}_\beta$ ((4.54)-(4.56)) to be T^n equivariant. Further it is clear that $m_{l,\beta} = 0$ for $\mathcal{M}^{\text{main}}_{l+1}(L(p), \beta)^{\mathfrak{s}_\beta} = \emptyset$.
In order to check that the A_∞-relation can be derived following this approach, recall equation (4.80)

$$\underbrace{d((\mathcal{M}, \mathfrak{s}, ev_{s,t})_*(\cdot))}_{=0} = \underbrace{(\mathcal{M}, \mathfrak{s}, ev_{s,t})_*(d\cdot)}_{=0} \pm (\partial\mathcal{M}, \mathfrak{s}, ev_{s,t})_*(\cdot) . \tag{5.108}$$

The two indicated terms vanish since we are working with harmonic and thus especially closed forms. We already outlined how $\partial\mathcal{M}$ looks like. With (5.91) and (5.108) we get

$$\begin{aligned} (\bigcup_{l_1+l_2=l+1} \bigcup_{\beta_1+\beta_2=\beta} \bigcup_{j=1}^{l_2} \overbrace{\mathcal{M}^{\text{main}}_{l_1+1}(L(p), \beta_1)^{\mathfrak{s}_{\beta_1}}_{ev_0} \times_{ev_j} \mathcal{M}^{\text{main}}_{l_2+1}(L(p), \beta_2)^{\mathfrak{s}_{\beta_2}}}^{=\mathcal{M}_{rt}}, \\ \mathfrak{s}_{rt}, ev_{rt,r}, ev_{rt,t})_*(\alpha_1 \times ... \times \alpha_l) = 0 . \end{aligned} \tag{5.109}$$

Recall Lemma 4.3 namely how the transport of forms is composed and thereof get

$$\begin{aligned} \sum_{l_1+l_2=l+1} \sum_{\beta_1+\beta_2=\beta} \sum_{j=1}^{l_2} (-1)^{\diamond_l} (\mathcal{M}^{\text{main}}_{l_2+1}(L(p), \beta_2)^{\mathfrak{s}_{\beta_2}}, \mathfrak{s}_{st}, \overbrace{(ev_j, ..., ev_{j+l-l_1})}^{ev_{st}}, \overbrace{ev_0}^{ev_t})_* \circ \\ \circ(\mathcal{M}^{\text{main}}_{l_1+1}(L(p), \beta_1)^{\mathfrak{s}_{\beta_1}}, \mathfrak{s}_{rs}, \underbrace{(ev_1, ..., \widehat{(ev_{st})}, ..., ev_l)}_{ev_r}, \underbrace{ev_0}_{ev_{rs}})_*(\alpha_1 \times ... \times \alpha_l) = 0. \end{aligned} \tag{5.110}$$

Here the sign prefactor

$$\diamond_l := \sum_{i=1}^{j-1} (\deg\, \alpha_i + 1) \tag{5.111}$$

seems to be chosen a bit arbitrarily in a way such that everything fits nicely into the desired A_∞-algebra concept. In fact the reader is referred to chapter 8 of [FOOO1] where the orientation issues respectively sign conventions are discussed. There it is proven that one can find orientations (depending on the relative spin structure on $L(p)$) that fit appropriately in the way described here.

According to the manner how $m_{l,\beta}$ is defined, equation (5.110) can be rewritten as

$$\sum_{l_1+l_2=l+1} \sum_{j=1}^{l_2} \sum_{\beta_1+\beta_2=\beta} (-1)^{\diamond_l} m_{l_1,\beta_1}(\alpha_1, ..., \alpha_{j-1}, m_{l_2,\beta_2}(\alpha_j, ..., \alpha_{j+l-l_1}), ..., \alpha_l) = 0 \tag{5.112}$$

Theorem 5.3

The data

$$(H^*(L(p), \Lambda_0^{\mathbb{R}}), m_l, \mathcal{F}) := (H^*(L(p), \mathbb{R}) \otimes_{\mathbb{R}} \Lambda_0, \sum_{\beta \in \pi_2(M, L(p))} T^{\frac{\omega(\beta)}{2\pi}} m_{l,\beta}, \mathcal{F})$$

defines a filtered A_∞-algebra ($\mathcal{F}$ an energy filtration) carrying the following properties:

(i) being weak, that is

$$m_0(1) \in H^0(L(p), \Lambda_0^{\mathbb{R}}) \tag{5.113}$$

not necessarily equal 0

(ii) being G-gapped, that is

$$G = (\underbrace{\{\omega(\beta)/2\pi \mid m_{l,\beta} \neq 0\}}_{\subset \mathbb{R}_{\geq 0}}, +, 0) \tag{5.114}$$

is a discrete submonoid and therefore satisfies (i) of condition 3.1

(iii) being unital with unit

$$\mathbf{e} = PD([L(p)]) \in H^0(L(p), \Lambda_0^{\mathbb{R}}) \tag{5.115}$$

given by the Poincaré dual of the fundamental class of $L(p)$.

Proof: Recall section 2.1 namely that Λ_0 is a principal ideal domain and thus the universal coefficient theorem allows to just tensor it with $H^*(L(p), \mathbb{R})$ in order to get a graded $\Lambda_0^{\mathbb{R}}$ module.

Obviously the filtration $\mathcal{F}$ induced by

$$T^{\lambda} H^{l}(L(p), \Lambda_0^{\mathbb{R}}) \quad \text{for} \quad \lambda \in \mathbb{R} \tag{5.116}$$

is an energy filtration satisfying axioms $(i) - (v)$ of Def. 3.3. Remark that

$$\Lambda_0^{\mathbb{R}} \cong \frac{\Lambda_{0,nov}^{\mathbb{R}}}{(e-1)} \tag{5.117}$$

that is forgetting the formal parameter e further guarantees convergence of m_k and therefore im $(m_k) \in H^*(L(p), \Lambda_0^{\mathbb{R}})$.
The A_∞-relation (3.63) is satisfied by (5.112), namely one has

$$\begin{aligned}
&\sum_{l_1+l_2=l+1} \sum_{j=1}^{l_2} (-1)^{\diamond_l} m_{l_1}(\alpha_1, ..., \alpha_{j-1}, m_{l_2}(\alpha_j, ..., \alpha_{j+l-l_1}), ..., \alpha_l) = \\
&= \sum_{l_1+l_2=l+1} \sum_{j=1}^{l_2} \sum_{\beta_1+\beta_2=:\beta} (-1)^{\diamond_l} T^{\frac{\omega(\beta_1)+\omega(\beta_2)}{2\pi}} \\
&\qquad\qquad m_{l_1,\beta_1}(\alpha_1, ..., \alpha_{j-1}, m_{l_2,\beta_2}(\alpha_j, ..., \alpha_{j+l-l_1}), ..., \alpha_l) = \\
&= \sum_{\beta} T^{\frac{\omega(\beta)}{2\pi}} \sum_{l_1+l_2=l+1} \sum_{j=1}^{l_2} \sum_{\beta_1+\beta_2=\beta} (-1)^{\diamond_l} \\
&\qquad\qquad m_{l_1,\beta_1}(\alpha_1, ..., \alpha_{j-1}, m_{l_2,\beta_2}(\alpha_j, ..., \alpha_{j+l-l_1}), ..., \alpha_l) = 0.
\end{aligned} \tag{5.118}$$

In order to check (i) we need a reasonable form for $m_{0,\beta}$. This yields that m_0 is of the form

$$m_0(1) = \sum_{\beta} T^{\frac{\omega(\beta)}{2\pi}} ev_*([\mathcal{M}_1^{\text{main}}(L(p), \beta)^{\mathfrak{s}_\beta}]) = \sum_{\mu(\beta)=2} T^{\frac{\omega(\beta)}{2\pi}} \underbrace{[L(p)]}_{\in H_n(L(p),\mathbb{R}) \cong H^0(L(p),\mathbb{R})} . \tag{5.119}$$

Degree considerations imply that in the sum we just have to consider homotopy classes β with Maslov index $\mu(\beta) = 2$. We soon describe this aspect a bit more precisely in (5.131) and (5.134). Fact (ii) and (iv) of Proposition 5.3 then imply

$$L(p) \cong \mathcal{M}_1^{\text{main}}(L(p), \beta_i) = \mathcal{M}_1^{\text{main}}(L(p), \beta_i)^{\mathfrak{s}_{\beta_i}} = \mathcal{M}_1^{\text{main}}(L(p), \beta)^{\mathfrak{s}_\beta} \tag{5.120}$$

meaning that the usage of multisections is redundant here since we already have a manifold structure.

G is a submonoid with identity $\omega(\beta_0 = 0)/2\pi = 0$ (β_0 represented by constant maps) since for the symplectic volume one has $\omega(\beta_1) + \omega(\beta_2) = \omega(\beta_1 + \beta_2)$. The fact that it is discrete in $\mathbb{R}_{\geq 0}$ follows by (vi) of of Proposition 5.3, namely

$$\omega(\beta) = \sum_{i=1}^{N} \underbrace{k_i}_{\in \mathbb{Z}_{\geq 0}} \omega(\beta_i) + \sum_j \omega(\alpha_j) \ . \tag{5.121}$$

The fact that $\omega(\alpha_j) \in \mathbb{Z}_{\geq 0}$ follows by Gromov´s compactness theorem namely it states amongst others that

$$\{\omega(\alpha) \mid \exists \text{ holomorphic sphere } w \text{ of class } \alpha\} \tag{5.122}$$

is a discrete subgroup (and therefore $Z_{\geq 0}$) of $\mathbb{R}_{\geq 0}$.

For assertion (iii) we make use of Lemma 7.3.2. of [FOOO1]. Since in our case due to Lemma 11.2. of [FOOO2] we have

$$(\mathfrak{forget}_0)^* \mathfrak{s}_{\beta,1} = \mathfrak{s}_{\beta,k+1} \tag{5.123}$$

The stated Lemma provides the identity

$$[m_{l,\beta}(\alpha_1, ..., \alpha_{j-1}, \mathbf{e}, \alpha_{j+1}, ..., \alpha_l)] = \begin{cases} [0], & (l,\beta) \neq (l, \beta_0 = 0) \\ (-1)^{\deg(\alpha_1)}[\alpha_1], & (l,\beta) = (1,\beta_0) \end{cases} \tag{5.124}$$

that yields the required relations 3.5 (d):

$$[m_l(...)] = \sum_{\beta \in \pi_2(M, L(p))} T^{\frac{\omega(\beta)}{2\pi}} [m_{l,\beta}(...)] = (-1)^{\deg(\alpha_1)} T^{\frac{\omega(\beta_0)}{2\pi}} [\alpha_1] = (-1)^{\deg(\alpha_1)}[\alpha_1]. \tag{5.125}$$ ■

Remark 5.1. *Using $\Lambda^{\mathbb{R}}_{0,nov}$ instead of $\Lambda^{\mathbb{R}}_0$ coefficients and defining*

$$m_l := \sum_{\beta \in \pi_2(M, L(p))} T^{\frac{\omega(\beta)}{2\pi}} e^{\frac{\mu(\beta)}{2}} m_{l,\beta} \tag{5.126}$$

provides that m_l is a degree $+1$ map. Recall that $m_{l,\beta}$ is of degree $1 - \mu(\beta)$, the formal generator e is of degree 2 and since we are considering oriented Lagrangian tori $L(p)$ the Maslov index is in $2\mathbb{Z}$. We resign this degree advantage since we want to utilize the better ring theoretical handling (principal ideal domain etc.) of $\Lambda^{\mathbb{R}}_0$ than $\Lambda^{\mathbb{R}}_{0,nov}$.

The way how to define a Lagrangian Floer cohomology

$$HF(H^*(L(p), \Lambda^{\mathbb{R}}_0), b; \Lambda^{\mathbb{R}}_{0,nov}) \equiv HF(L, b; \Lambda^{\mathbb{R}}_0) \tag{5.127}$$

now is already presented in section 3.2.2 (usually we follow the weak approach since it is more general). To do so we have to deal with weak Maurer-Cartan solutions b that are related to the potential function $\mathfrak{PO}$ as they form its domain. As the headline of section 5.5 suggests we are aiming to explicitly calculate $\mathfrak{PO}$ for toric fibers out of the previously defined A_∞-algebra structure.
For the definition of $\mathfrak{PO}$ recall (3.126). In our case it is of the form

$$\begin{aligned} &\mathfrak{PO} : \widehat{\mathcal{M}}_{\text{weak}}(H^*(L(p), \Lambda^{\mathbb{R}}_0)) \to \Lambda^+_0(\mathbb{R}) \equiv \Lambda^{\mathbb{R}}_+ \\ &\mathfrak{PO}(b) \text{ defined by } m(e^b) = \mathfrak{PO}(b) \cdot \mathbf{e} \end{aligned} \tag{5.128}$$

for $b \in \widehat{\mathcal{M}}_{\text{weak}}(H^*(L(p), \Lambda^{\mathbb{R}}_0)) \subset H^1(L(p), \Lambda^{\mathbb{R}}_0)$.
To be consistent with the literature we remark that $\mathcal{M}_{\text{weak}}(\cdots)$ arises of $\widehat{\mathcal{M}}_{\text{weak}}(\cdots)$ and by identifying *gauge equivalent* elements b_0, b_1 of the latter. The reader is referred to Definition 4.3.19. of [FOOO1] where this equivalence relation $\sim_G$ is

defined and properly discussed in the ongoing of the stated chapter 4. Here we only remark that it is possible to show that $\sim_G$ is trivial ($b_0 \sim_G b_1 \Rightarrow b_0 = b_1$) on $H^1(L(p), \Lambda_0^{\mathbb{R}})$ in our toric setup and we thus can forget the $\widehat{}$ sign meaning that we write

$$\widehat{\mathcal{M}}_{\text{weak}}(\cdots) \equiv \mathcal{M}_{\text{weak}}(\cdots) \tag{5.129}$$

likewise for weak Maurer-Cartan solutions in the following.
To achieve more transparency of the potential´s nature, we outline that for such a weak Maurer-Cartan solution b we have

$$\begin{aligned} m(e^b) &= m(1 + b \otimes b + b \otimes b \otimes b + ...) = \\ &= \sum_{l=0}^{\infty} m_k(\underbrace{b, ..., b}_{l}) = \\ &= \sum_{l} \sum_{\beta} T^{\frac{\omega(\beta)}{2\pi}} m_{l,\beta}(b, ..., b) \overset{!}{\sim} \underbrace{PD([L(p)])}_{=\mathbf{e}} . \end{aligned} \tag{5.130}$$

We have to find a way how to compute the above proportionality factor. For this we need an explicit expression for $m_{l,\beta}$. The next section´s considerations show that the homomorphisms $m_{l,\beta}$ (and therefore also $\mathfrak{PO}$) can be expressed by purely 'coordinate' data of the corresponding polytope in $\mathbb{R}^n$ (the symplectic volume is also computable in terms of polytopial data).
First remark that according to their definition we have

$$\begin{aligned} \deg(m_{l,\beta}(b, ..., b)) &= \sum_{i=1}^{l} 1 + \dim L_t - \text{vir.dim } \mathcal{M}_{l+1}^{\text{main}}(L(p), \beta)^{\mathfrak{s}_\beta} = \\ &= l + n - (n + \mu(\beta) + l + 1 - 3) = 2 - \mu(\beta) \overset{!}{\geq} 0 . \end{aligned} \tag{5.131}$$

Therefore in the sum (5.130) we only need to consider homology classes β with Maslov index

$$\mu(\beta) \in \{2\} \cup 2\mathbb{Z}_{\leq 0} . \tag{5.132}$$

Using the ideas of chapter 5.2 we conclude that for the calculation of $m(e^b)$ the case $\mu \leq 0$ can also be excluded. More precisely one purpose of perturbing moduli spaces by using multisections $\mathfrak{s}_\beta$ was to make

$$ev : \mathcal{M}_1^{\text{main}}(L(p), \beta)^{\mathfrak{s}_\beta} \to L(p) \tag{5.133}$$

submersive. By comparing dimensions

$$n = \dim \text{L(p)} \leq \text{vir.dim } \mathcal{M}_1^{\text{main}}(L(p), \beta)^{\mathfrak{s}_\beta} = n + \mu(\beta) + 1 - 3 \tag{5.134}$$

we conclude $\mu(\beta) \geq 2$ respectively $\mu(\beta) = 0$ for the class of constant curves $\beta = \beta_0$. In the proof of the upcoming proposition 5.5 we thus only have to consider curves of these two specific types. Thanks to the work of C.-H. Cho and Y.-G. Oh in [CO] (a short idea providing summary is given in section 5.4), pseudo-holomorphic discs representing these allowed classes β can be nicely described by 'coordinate' data of the underlying moment polytope.

Proposition 5.5

For a toric manifold M^{2n} the potential function $\mathfrak{PO}$ is defined for degree one cohomology classes b with strictly positive valuation value $\nu(b) \in (0, \infty)$. That is we have an embedding

$$H^1(L(p), \Lambda_+^{\mathbb{R}}) \hookrightarrow \widehat{\mathcal{M}}_{weak}(H^*(L(p), \Lambda_0^{\mathbb{R}})) \ . \tag{5.135}$$

The potential when restricted to $H^1(L(p), \Lambda_+)$ can then written as

$$\begin{aligned} \mathfrak{PO}(b) = \mathfrak{PO}(x_1, ..., x_n; p_1, ..., p_n) &= \\ = \sum_{i=1}^{N} y_1^{v_{i,1}} ... y_n^{v_{i,n}} T^{l_i(p)} &+ \\ &+ \underbrace{\sum_j c_j y_1^{v'_{j,1}} ... y_n^{v'_{j,n}} T^{l'_j(p)+\rho_j}}_{=\ 0 \ \text{for } M \text{ Fano}} \end{aligned} \tag{5.136}$$

Before giving a proof of the statements above and thereby a depiction of the appearing variables (see chapter 5.5), we first try to enlighten the abstract build up by making use of the advantage that we are dealing with toric manifolds. This fact simplifies the handling of the appearing pseudo-holomorphic curves since relevant ingredients (sympletic volume etc.) can be described by coordinate data.

5.4 Characterization of holomorphic discs via polytopial data

As motivated in the end of the last section we aim to get more insight into the nature of the homomorphisms $m_{l,\beta}$. In order to do so we luckily can refer to the work of C.-H. Cho and Y.-G. Oh in [CO]. Their work provides a possibility to describe holomorphic discs, of class $\beta \in \pi_2(M, L)$ attaching the Lagrangian torus fibers L, in a more seizable way. Especially we get an expression for their symplectic energy $\omega(\beta)$ by coordinate data of the corresponding moment polytope. This description works at least for β with Maslov index $\mu(\beta) = 2$. As already indicated above, by degree considerations, these classes are enough to be understood for our purpose, that is to prove Proposition 5.5. To see how to interpret the results of [CO] we first have to review and deepen some facts of how the construction of n dimensional toric manifolds out of fans respectively their dual polytopes is performed.
As indicated in (5.10) polytopes can be described by intersections of N half-spaces (with weights λ_i) of dimension n, and those in turn by the inward pointing normal vectors $v_i \in \mathfrak{t}$ $(i \in \{1, ..., d\})$. We try to put this in a more general context here. For a detailed description (and especially proofs) we again refer to [Au].

Linear independent vectors $v_i \in \mathbb{Z}^n =: N$ form *integral generators* of convex subsets called *k-dimensional cones* defined as

$$\sigma^{(k)} \equiv \sigma := \underbrace{\{a_1 v_1 + ... + a_k v_k \mid a_i \in \mathbb{R}_{\geq 0}\}}_{\text{abbreviated by } \langle v_1,...,v_k\rangle_0} \subset N \otimes \mathbb{R} =: N_\mathbb{R} \ . \tag{5.137}$$

To relate this point of view with the concepts presented in the beginning of chapter 5.1, think about these vectors v_i as the inward pointing normal vectors of the moment polytope.
A cone σ' of the form

$$\sigma' := \{a_{i_1} v_{i_1} + ... + a_{i_l} v_{i_l} \mid a_{i_j} \in \mathbb{R}_{\geq 0}\} \quad \text{for} \quad l \leq k. \tag{5.138}$$

is called a *l-dimensional face* of $\sigma = \langle v_1, ..., v_k\rangle_0$ (expressed by $\sigma' \prec \sigma$).
A collection Σ of such cones $\sigma_1, ..., \sigma_s$ is called a *complete n dimensional fan* if the following conditions hold for $i \in \{1, ..., s\}$:

(i) $\sigma_i \in \Sigma \ , \ \sigma_i' \prec \sigma_i \ \Rightarrow \ \sigma_i' \in \Sigma$

(ii) $\sigma_i' \cap \sigma_i \prec \sigma_i \ , \ \sigma_i' \cap \sigma_i \prec \sigma_i'$

(iii) $\bigcup_{i=1}^{s} \sigma_i = N_\mathbb{R}$.

Further denote by $\Sigma^{(k)} := \{\sigma_i^{(k)}\} \subset \Sigma$ the set of k dimensional cones in Σ.
Vectors generating cones of dimension one in $N_\mathbb{R}$

$$\{(v_1, ..., v_{N:=|\Sigma^{(1)}|}) \mid \langle v_i\rangle_0 \in \Sigma^{(1)}\} =: G(\Sigma) \tag{5.139}$$

are used to map $N_\mathbb{R}$ isomorphically to $\mathbb{C}^N$ via $v_i \leftrightarrow z_i$, for $\{z_1, ..., z_N\}$ being a basis of $\mathbb{C}^N$ (do not mix up the number N with the set $N = \mathbb{Z}^n$). Additionally we identify subsets

$$\begin{aligned} \{(v_{i_1}, ..., v_{i_p}) \mid \langle v_{i_1}, ..., v_{i_p}\rangle_0 \neq \Sigma^{(p)}, \langle v_{i_1},..., v_{i_k}\rangle_0 \in \Sigma^{(k)} \ \forall \ 0 \leq k < p\} = \\ =: \mathcal{P} \subset G(\Sigma) \\ \updownarrow \\ \{z \mid z_{i_1} = ... = z_{i_p} = 0\} =: \mathbb{A}(\mathcal{P}) \subset \mathbb{C}^N \ . \end{aligned} \tag{5.140}$$

For a cone $\sigma^{(k)} = \langle v_{i_1}, ..., v_{i_k}\rangle_0$ generated by $\{v_{i_1}, ..., v_{i_k}\}$ we further define subsets

$$U(\sigma^{(k)}) := \underbrace{\{z \mid z_j \neq 0 \text{ for } j \notin \{i_1, ..., i_k\}\}}_{\cong \ \mathbb{C}^k \times (\mathbb{C}^*)^{N-k}} \subset \mathbb{C}^N \ . \tag{5.141}$$

One can easily check that for such sets $U(\sigma)$ the following holds:

(i) $U(\Sigma) := \bigcap_{\mathcal{P}} (\mathbb{C}^N - \mathbb{A}(\mathcal{P})) \equiv \bigcup_{\sigma\in\Sigma} U(\sigma)$

(ii) $\sigma' \prec \sigma \ \Rightarrow \ U(\sigma') \subset U(\sigma)$

(iii) $U(\sigma') \cap U(\sigma) = U(\sigma' \cap \sigma)$.

The orbit spaces (in fact they are manifolds since, as we see soon, the action of $D(\Sigma)$ is free)

$$X_\Sigma := U(\Sigma)/D(\Sigma) \quad \text{and} \quad X_{\sigma^{(n)}} := U(\sigma^{(n)})/D(\Sigma) \tag{5.142}$$

are now defined to be the *compact toric n dimensional manifold associated with* Σ respectively their *charts* covering it ($U(\Sigma) = \bigcup_{\sigma\in\Sigma} U(\sigma)$).
So let us clarify how the group action of $D(\Sigma)$ looks like and especially why this action is considered to be free.
The connected commutative subgroup (componentwise multiplication)

$$D(\Sigma) \subset (\mathbb{C}^*)^N \tag{5.143}$$

is generated by elements

$$(t^{\lambda_1}, ..., t^{\lambda_N}) \quad \text{for} \quad t \in \mathbb{C}^* \ . \tag{5.144}$$

Here $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{Z}^N$ are of the form such that

$$\lambda_1 v_1 + ... + \lambda_N v_N = 0 \tag{5.145}$$

for the vectors v_i as in (5.139). It is clear that $D(\Sigma)$ acts freely on $U(\Sigma)$ respectively $U(\sigma)$ by componentwise multiplication.
We get that the charts $X_{\sigma^{(n)}}$ induced by a cone of dimension n

$$\sigma^{(n)} = \langle v_{i_1}, ..., v_{i_n}\rangle_0 \tag{5.146}$$

($\{v_{i_1}, ..., v_{i_n}\}$ basis of $N = \mathbb{Z}^n$) are isomorphic to $\mathbb{C}^n$. Coordinates $(x_1, ..., x_n)$ can explicitly be written as

$$x_j^{\sigma^{(n)}} = z_1^{\langle v_1, u_{i_j}\rangle} \cdot ... \cdot z_N^{\langle v_N, u_{i_j}\rangle} \ . \tag{5.147}$$

Here $\{u_{i_1}, ..., u_{i_n}\}$ denotes the basis of

$$M := Hom_{\mathbb{Z}}(N, \mathbb{Z}) \tag{5.148}$$

dual to $\{v_{i_1}, ..., v_{i_n}\}$.
The modalities of how one constructs the explicit torus action and the symplectic form on X_Σ can be found [Au]. We skip this point here since we do not need its specific form in the upcoming progress.

For not just stating abstract theoretical concepts we want to get a bit more specific and illustrate how they are applied for constructing the toric manifold $\mathbb{C}P^2$ (see example b) in section 5.1) out of given 'fan' data.

Example: From Σ to $\mathbb{C}P^2$

Recall the moment polytope (5.31) for $\mathbb{C}P^2$. For the 2 dimensional fan Σ we get

$$G(\Sigma) = \{v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1)\} \quad \text{that is} \quad v_i \in \mathbb{Z}^2 \ . \tag{5.149}$$

These 3 inward pointing normal vectors are pairwise linearly independent in $\mathbb{R}^2$ thus generate 2 dimensional cones. Since we can not have 3 dimensional cones we get

$$\mathcal{P} = \{v_1, v_2, v_3\} \longleftrightarrow \mathbb{A}(\mathcal{P}) = \{0\} \subset \mathbb{C}^3. \tag{5.150}$$

This implies that

$$U(\Sigma) = \mathbb{C}^3 - \{0\} = (\mathbb{C}^*)^3 \ . \tag{5.151}$$

The equation

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \tag{5.152}$$

is solved by $\{(a,a,a) \mid a \in \mathbb{Z}\}$ and therefore

$$D(\Sigma) = \{(z,z,z) \mid z \in \mathbb{C}^*\} \cong \mathbb{C}^* \tag{5.153}$$

consists of the diagonal elements in $(C^*)^3$. So

$$X_\Sigma = U_\Sigma / D_\Sigma = (\mathbb{C}^*)^3 / \mathbb{C}^* \tag{5.154}$$

is in fact the 2 dimensional complex projective space $\mathbb{C}P^2$.
For charts $X_{\sigma_{ij}^{(2)}}$ we regard the 2 dimensional cones $\sigma_{ij}^{(2)}$ and identify them as described in (5.142):

$$\begin{aligned}
\sigma_{23}^{(2)} &= \langle v_2, v_3 \rangle_0 \longrightarrow U(\sigma_{23}^{(2)}) = \{z \in \mathbb{C}^3 \mid z_1 \neq 0\} \longrightarrow X_{\sigma_{23}^{(2)}} = \{[z_1, z_2, z_3] \mid z_1 \neq 0\} \\
\sigma_{13}^{(2)} &= \langle v_1, v_3 \rangle_0 \longrightarrow U(\sigma_{13}^{(2)}) = \{z \in \mathbb{C}^3 \mid z_2 \neq 0\} \longrightarrow X_{\sigma_{13}^{(2)}} = \{[z_1, z_2, z_3] \mid z_2 \neq 0\} \\
\sigma_{12}^{(2)} &= \langle v_1, v_2 \rangle_0 \longrightarrow U(\sigma_{12}^{(2)}) = \{z \in \mathbb{C}^3 \mid z_3 \neq 0\} \longrightarrow X_{\sigma_{12}^{(2)}} = \{[z_1, z_2, z_3] \mid z_3 \neq 0\}
\end{aligned} \tag{5.155}$$

These are the well know charts covering $\mathbb{C}P^2$. Using (5.147) they are mapped to $\mathbb{C}^2$ also in the common fashion, namely coordinates for e.g. $X_{\sigma_{12}^{(2)}}$ are of the form

$$\begin{aligned}
x_1^{\sigma_{12}^{(2)}} &= z_1^{\langle v_1, e_1 \rangle} \cdot z_2^{\langle v_2, e_1 \rangle} \cdot z_3^{\langle v_3, e_1 \rangle} = z_1 \cdot z_3^{-1} \\
x_2^{\sigma_{12}^{(2)}} &= z_1^{\langle v_1, e_2 \rangle} \cdot z_2^{\langle v_2, e_2 \rangle} \cdot z_3^{\langle v_3, e_2 \rangle} = z_2 \cdot z_3^{-1} \ .
\end{aligned} \tag{5.156}$$

Characterization of holomorphic discs

We aim to state some results of [CO]. These are necessary in order find coordinate expressions (in the sense of (5.147)) for holomorphic discs of class β_i with Maslov index two.
As we will see, there are only $N = |G(\Sigma)|$ discs of this type and we actually can compute their symplectic volume

$$\omega(\beta_i) \quad \text{for} \quad i \in \{1, ..., N\} \tag{5.157}$$

in terms of the inward pointing normal vectors v_i and the conditions λ_i, determining the half-spaces of Δ in (5.10).
For a holomorphic disc

$$w : (D^2, \partial D^2) \to (X_\Sigma, L(p)) \quad \text{of class} \quad w_*([D^2]) = \beta \tag{5.158}$$

attaching the Lagrangian subtorus $L(p) = \mu^{-1}(p)$ (for $p \in \mathring{\Delta}$), the Maslov index can be computed by

$$\mu(\beta) = 2 \cdot (\text{intersection multiplicities of } \mathrm{im}(w) \text{ with } V(v_i)) \ . \tag{5.159}$$

The $n-1$ dimensional manifolds $V(v_i)$ are defined by the image

$$\pi(\{z \in U(\Sigma) \mid z_i = 0\}) \tag{5.160}$$

for π being the projection of the principal bundle

$$U(\Sigma) \xrightarrow{\pi} U(\Sigma)/D(\Sigma) = X_\Sigma \ . \tag{5.161}$$

One can show that $V(v_i) = \mu^{-1}(\partial\Delta_i)$ for μ being the moment map and $\partial\Delta_i$ the i-th face as in (5.88). To find coordinate expressions for such discs we make use of the fact (Theorem 5.3. of [CO]) that holomorphic discs can be lifted ($\pi \circ \widetilde{w} = w$) in the following way:

$$\begin{tikzcd} & \left(\bigcap_{\mathcal{P}}(\mathbb{C}^N - \mathbb{A}(\mathcal{P})), \pi^{-1}(L(p))\right) \arrow[d, "\pi"] \\ (D^2, \partial D^2) \arrow[ur, "\widetilde{w}"] \arrow[r, "w"'] & (X_\Sigma, L(p)) \end{tikzcd}$$

Further developing the argumentation of this classification theorem illustrates that holomorphic discs $D(v_1), ..., D(v_N)$ of the form

$$D(v_i) = \pi(\{\widetilde{w}(z) \in U(\Sigma) \mid z_l = c_l \text{ for } l \neq i, \ z_i = c_i \cdot z, \ z \in D^2\}) \tag{5.162}$$

and of class $\beta_1, ..., \beta_N$ are the only ones (up to reparametrization by $\mathrm{PSL}(2, \mathbb{C})$) with Maslov index $\mu(\beta_i) = 2$. Proposition 7.4. of [CO] additionally provides that such discs $D(v_{i_1}), ..., D(v_{i_n})$ are contained in the chart $X_{\sigma^{(n)}}$ obtained by the cone

$$\sigma^{(n)} = \langle v_{i_1}, ..., v_{i_n} \rangle_0 \ . \tag{5.163}$$

We can use (5.147) and write an explicitely coordinate expression for a disc

$$D(v_i) \subset X_{\sigma^{(n)}} \tag{5.164}$$

in terms of coordinates of $\mathbb{C}^n$, namely

$$(c'_1 \cdot z^{v_{i_1}}, ..., c'_n \cdot z^{v_{i_n}}) \ . \tag{5.165}$$

Here $z \in D^2$ and the c'_l are constants in $\mathbb{C}^*$ chosen such that the boundary of the discs attach the Lagrangian submanifold $\partial D(v_i) \subset L$.

In Theorem 8.1. of [CO] C.-H. Cho and Y.-G. Oh actually compute the symplectic volume $\omega(\beta_i)$ of these specific discs. For $D(\beta_i)$ attaching the Lagrangian torus fiber $L(p)$ it is given by

$$\omega(\beta_i) = 2\pi(\langle p, v_i\rangle - \lambda_i) =: 2\pi l_i(p) \ . \tag{5.166}$$

As claimed we now have an expression that allows to compute the symplectic energy of all holomorphic discs of Maslov index 2 in terms of the corresponding polytopial data.

Interrelation between Fans and (Co-)Homology

The last ingredient for proving Proposition 5.5 is to achieve a coordinate description $(x_1, ..., x_n)$ for weak bounding cochains

$$b \in H^1(L(p); \Lambda_+^{\mathbb{R}}) \ . \tag{5.167}$$

This further allows to link the cohomology/homology pairing $\partial\beta_i(b)$ with the in (5.136) appearing product

$$\prod_{j=1}^{n} y_j^{v_{i_j}} \tag{5.168}$$

in (5.136) for $i \in \{1, ..., N\}$.
Cones for a given fan live in $N_{\mathbb{R}}$ which is characterized by the lattice

$$N = \mathbb{Z}^n \cong H_1(T^n; \mathbb{Z}) \cong H_1(L(p); \mathbb{Z}) \tag{5.169}$$

since the n dimensional torus T^n and the torus fiber $L(p)$ are diffeomorphic.
So by choosing an integral basis $\{e_1^*, ..., e_n^*\}$ for $N = \mathbb{Z}^n$ and dualizing it, provides $\{e_1, ..., e_n\}$ as a basis for $H^1(L(p); \mathbb{Z})$. This in turn allows to decompose

$$H^1(L(p); \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda_+^{\mathbb{R}} \cong H^1(L(p); \Lambda_+^{\mathbb{R}}) \ni b = \sum_{j=1}^{n} x_j e_j \tag{5.170}$$

with coefficients $x_j \in \Lambda_+^{\mathbb{R}}$. We additionally define

$$y_j := e^{x_j} = \sum_{k=0}^{\infty} \frac{x_j^k}{k!} \in \Lambda_0^{\mathbb{R}} \ . \tag{5.171}$$

For now interpreting $\partial\beta_i(b)$ for a disc $D(v_i)$ of class $w_*[D(v_i)] = \beta_i$ (for $i \in \{1, ..., N\}$) we make use of (5.165) that provides a description of $\partial D(v_i)$ by

$$(c_1' \cdot z^{v_{i_1}}, ..., c_n' \cdot z^{v_{i_n}}) \quad \text{with} \quad |z| = 1. \tag{5.172}$$

For the boundary class $\partial\beta_i \in H_1(L(p); \mathbb{Z})$ and $b = \sum_{j=1}^{n} x_j e_j \in H^1(L(p); \Lambda_+^{\mathbb{R}})$ we can therefore describe its pairing by using the described dual bases

$$\partial\beta_i(b) = \sum_{j=1}^{n} v_{i_j} x_j = \langle v_i, b\rangle \ . \tag{5.173}$$

In the upcoming proof this allows to use the identity

$$\sum_{k=0}^{\infty} \frac{\partial \beta_i(b)}{k!} = e^{\langle v_i, b\rangle} = \prod_{j=1}^{n} y_j^{v_{i_j}}. \tag{5.174}$$

5.5 Explicite Calculation of $\mathfrak{PO}$ for Toric Manifolds (proof of Proposition 5.5)

The upcoming proof is a summary of the considerations presented in chapter 11 of [FOOO2]. In the following section we make use of $x \in H^1(L(p); \mathbb{R})$, $b \in H^1(L(p); \Lambda_+^{\mathbb{R}})$, $l \geq 0$ and $\beta \in \pi_2(M, L(p))$.

(i) Computation of $m_{l,\beta}$:

As a first step we try to calculate

$$\int_{L(p)} m_{l,\beta}(x, ..., x) \tag{5.175}$$

for relative homotopy classes β with Maslov index $\mu(\beta) = 2$.
As a working tool consider the compactified moduli space $\mathcal{M}_{l+1}(\mathbb{C}, S^1; \beta_{\mathbb{C}})$ of genus 0 stable maps

$$w : (\Sigma, \partial\Sigma) \to (\mathbb{C}, S^1) \tag{5.176}$$

of class $\beta_{\mathbb{C}} \in H_2(\mathbb{C}, S^1; \mathbb{Z})$ with $l+1$ marked points $\{z_0, ..., z_l\}$ from bordered Riemann surfaces Σ onto $\mathbb{C}$ attaching the Lagrangian submanifold $S^1 \subset \mathbb{C}$. The corresponding $l+1$ component evaluation map is of the form

$$\begin{aligned} ev = (ev_0, ..., ev_l) : \mathcal{M}_{l+1}(\mathbb{C}, S^1; \beta_{\mathbb{C}}) &\to (S^1)^{l+1} \\ [w; (z_0, ..., z_l)] &\mapsto (w(z_0), ..., w(z_l)) \ . \end{aligned} \tag{5.177}$$

For a fixed base point $p_0 \in S^1$ we have

$$\mathcal{M}_{l+1}(\mathbb{C}, S^1; \beta_{\mathbb{C}})_{p_0} := ev_0^{-1}(p_0) \subset \mathcal{M}_{l+1}(\mathbb{C}, S^1; \beta_{\mathbb{C}}) \tag{5.178}$$

as the subset with 'fixed' 0-th marked point. Due to (5.93) we know that $\mathfrak{s}_{\beta,k+1}$ arises as a pullback of $\mathfrak{s}_{\beta,1}$ by the map $\mathfrak{forget}_0$, forgetting all but the 0-th marked point. In that sense we get a fibration

$$\mathcal{M}_{l+1}^{\mathrm{main}}(L(p), \beta)^{\mathfrak{s}_\beta} \to \mathcal{M}_{1}^{\mathrm{main}}(L(p), \beta)^{\mathfrak{s}_\beta} \tag{5.179}$$

with fiber $\mathcal{M}_{l+1}(\mathbb{C}, S^1; \beta_{\mathbb{C}})_{p_0}$. The information about the marked points $\{z_1, ..., z_l\}$ is encoded in $\mathcal{M}_{l+1}(\mathbb{C}, S^1; \beta_{\mathbb{C}})_{p_0}$ or more visible via the diffeomorphism

$$\begin{aligned} \mathcal{M}_{l+1}(\mathbb{C}, S^1; \beta_{\mathbb{C}})_{p_0} \cap \mathcal{M}_{l+1}^{\mathrm{reg}}(\mathbb{C}, S^1; \beta_{\mathbb{C}}) &\to \{(t_1, ..., t_l) \in \mathbb{R} \mid 0 < t_1 < ... < t_l < 1\} \\ [w; (z_0, ..., z_l)] &\mapsto (w(z_1) - w(z_0), ..., w(z_l) - w(z_0)) \ , \end{aligned} \tag{5.180}$$

which is well-defined since, due to (2.49), for curves of the same isomorphism class we have

$$w'(z_i') = w(\phi^{-1}(z_i')) = w(z_i) \ . \tag{5.181}$$

With (5.179) we get that the evaluation map

$$ev = (ev_0, ..., ev_l) : \mathcal{M}_{l+1}^{\text{main}}(L(p), \beta) \to L(p)^{l+1} \tag{5.182}$$

can be written as

$$\begin{aligned} ev_0(u; t_1, ..., t_k) &= ev(u) \\ ev_i(u; t_1, ..., t_k) &= [t_i \partial\beta] \cdot ev(u) \end{aligned} \tag{5.183}$$

for $u \in \mathcal{M}_1^{\text{main}}(L(p), \beta)$ and $\partial\beta \in H_1(L(p); \mathbb{Z})$.
Recall (5.105), namely the homomorphisms $m_{l,\beta}(\underbrace{x, ..., x}_{l})$, for $x \in H^1(L(p); \mathbb{R})$, are defined via pulling back l harmonic one forms of $L(p)$ with $(ev_1, ..., ev_l)$ and then integrating along the fiber

$$\mathcal{M}_1^{\text{main}}(L(p), \beta)^{\mathfrak{s}_\beta} \times \{(t_1, ..., t_l) \in \mathbb{R} \mid 0 < t_1 < ... < t_l < 1\} \tag{5.184}$$

with ev, which yields one $2 - \underbrace{\mu(\beta)}_{=2} = 0$ form on $L(p)$. With (5.183)

$$\int_{L(p)} m_{l,\beta}(x, ..., x) \tag{5.185}$$

thus writes as

$$\begin{aligned} \int_{L(p)} m_{l,\beta}(x, ..., x) &= \underbrace{\int_{L(p)} ev_*(\mathcal{M}_1^{\text{main}}(L(p), \beta)^{\mathfrak{s}_\beta})}_{=\ \deg\ [ev: \mathcal{M}_1^{\text{main}}(L(p),\beta) \to L(p)]\ =:\ c_\beta} \cdot \int_{t_1\partial\beta} x \ \cdots \int_{t_l\partial\beta} x = \\ &= c_\beta \underbrace{\text{vol}(\{(t_1, ..., t_l) \in \mathbb{R} \mid 0 < t_1 < ... < t_l < 1\})}_{=\ \frac{1}{l!}} \cdot (\int_{\partial\beta} x)^l = \frac{c_\beta}{l!} (\partial\beta(x))^l \end{aligned} \tag{5.186}$$

which is equivalent to

$$m_{l,\beta}(x, ..., x) = \frac{c_\beta}{l!} (\partial\beta(x))^l \cdot PD([L(p)]) \ . \tag{5.187}$$

(ii) Proof of $H^1(L(p), \Lambda_+^{\mathbb{R}}) \hookrightarrow \mathcal{M}_{\text{weak}}(H^*(L(p), \Lambda_0^{\mathbb{R}}))$:

For an element $b \in H^1(L(p), \Lambda_+^{\mathbb{R}})$ we have

$$m(e^b) = \sum_{l=0}^{\infty} \sum_{\beta} \underbrace{m_{l,\beta}(b, ..., b)}_{\in\ \Omega^{l-(n+\mu(\beta)+l+1-3)-n}(L(p))\ =\ \Omega^{2-\mu(\beta)}(L(p))} T^{\frac{\omega(\beta)}{2\pi}} \ . \tag{5.188}$$

Recall (5.131), namely the indicated degree consideration yield that we just have to sum over β with $\mu(\beta) = 0$ (with Proposition 5.3 (i) only for $\beta = \beta_0 = 0$) and β with $\mu(\beta) = 2$. For the case of constant curves (i.e. of class β_0) remark that for harmonic forms $\alpha \in \Omega^1(L(p))$, when following the ideas of the former proof we get

$$\begin{aligned} m_{0,\beta_0} &= 0 \\ m_{1,\beta_0}(\alpha) &= \pm d\alpha = 0 \\ m_{l\geq 2,\beta_0}(\alpha, ..., \alpha) &= \alpha \cup ... \cup \alpha \underbrace{=}_{\deg\,\alpha=1} 0 \ . \end{aligned} \tag{5.189}$$

Since only relative homotopy classes with Maslov index $\mu(\beta) = 2$ survive, we are allowed to apply the previous result (i) and thus write (5.188) as

$$\begin{aligned} m(e^b) &= \underbrace{\Big(\sum_{\substack{\beta, \\ \mu(\beta)=2}} \sum_{l=0}^{\infty} \frac{c_\beta}{l!} (\partial\beta(b))^l T^{\frac{\omega(\beta)}{2\pi}} \Big)}_{(*)} \cdot PD([L(p)]) = \\ &= \mathfrak{PO}(b) \cdot PD([L(p)]) \ . \end{aligned} \tag{5.190}$$

It is important that we require $b \in H^1(L(p), \Lambda_+^{\mathbb{R}})$ not having a constant summand (that is to work with $\Lambda_+^{\mathbb{R}}$ instead of $\Lambda_0^{\mathbb{R}}$ coefficients). It guarantees convergence (with respect to the energy filtration $\mathcal{F}$) of $(*)$, affecting in that the sums can be interchanged and that we can interpret the proportionality factor as the search potential function

$$\mathfrak{PO}(b) = \sum_{\substack{\beta, \\ \mu(\beta)=2}} \sum_{l=0}^{\infty} \frac{c_\beta}{l!} (\partial\beta(b))^l T^{\frac{\omega(\beta)}{2\pi}} \quad \in \Lambda_+^{\mathbb{R}} \ . \tag{5.191}$$

In summary we have that $H^1(L(p), \Lambda_+^{\mathbb{R}})$ embeds into $\mathcal{M}_{\text{weak}}(H^*(L(p), \Lambda_0^{\mathbb{R}}))$ and thus can be seen as a (not yet the whole) domain of $\mathfrak{PO}$.

(iii) <u>A coordinate expression for $\mathfrak{PO}$</u> :

Recall result (vi) of Proposition 5.3, namely

$$\beta = \sum_{i=1}^{N} k_i \cdot \underbrace{\beta_i}_{\substack{\text{holomorphic} \\ \text{disc}}} + \sum_j \underbrace{\alpha_j}_{\substack{\text{holomorphic} \\ \text{sphere}}} \tag{5.192}$$

and clearly

$$\partial\beta = \sum_{i=1}^{N} k_i \cdot \partial\beta_i \tag{5.193}$$

for β_i realized by the holomorphic discs analyzed in section (5.4). Considering the Maslov indices we know

$$2 = \mu(\beta) = \sum_{i=1}^{N} \underbrace{k_i}_{\in\ \mathbb{Z}_{\geq 0}} \cdot \underbrace{\mu(\beta_i)}_{=\ 2} + \sum_j \mu(\alpha_j) = 2\mathbb{Z}_{\geq 0} + \sum_j \mu(\alpha_j)\ . \tag{5.194}$$

Assume now that the corresponding compact toric symplectic manifold M is Fano which yields that every nontrivial holomorphic sphere $u : S^2 \to M$ of class α_j has positive Chern number

$$\langle c_1(M), \alpha_j \rangle > 0 \tag{5.195}$$

and therefore only nonnegative Maslov index can possibly appear in (5.194). Combining this with (5.191) we get that in the Fano case $\mathfrak{PO}(b)$ can be written as

$$\sum_{i=1}^{N} \sum_{l=0}^{\infty} \frac{c_{\beta_i}}{l!} (\partial\beta_i(b))^l T^{\frac{\omega(\beta_i)}{2\pi}}\ . \tag{5.196}$$

When recollecting our developed knowledge about holomorphic discs of class β_i (see section 5.4) we have:

(i) $\omega(\beta_i) = 2\pi l_i(p)$

(ii) $\partial\beta_i(b) = \langle v_i, b\rangle = \sum_{j=1}^{n} v_{i_j} x_j$

(iii) $ev : \mathcal{M}_1^{\text{main}}(L(p), \beta_i) \to L(p)$ is an orientation preserving diffeomorphism (Proposition 5.3 (iv)) and thus $c_{\beta_i} = \deg[ev : \mathcal{M}_1^{\text{main}}(L(p), \beta_i) \to L(p)] = 1$

In summary (which then also signifies the end of the proof) we get for M being Fano toric:

$$\boxed{\begin{aligned}\mathfrak{PO}(b) &= \sum_{i=1}^{N} \sum_{l=0}^{\infty} \frac{(\sum_{j=1}^{n} v_{i_j} x_j)^l}{l!} T^{l_i(p)} = \sum_{i=1}^{N} e^{v_{i_1} x_1} \cdot \ldots \cdot e^{v_{i_n} x_n} T^{l_i(p)} \underbrace{=}_{(5.171)} \\ &= \sum_{i=1}^{N} y_1^{v_{i_1}} \cdot \ldots \cdot y_n^{v_{i_n}} T^{l_i(p)}\end{aligned}} \tag{5.197}$$

For the non-Fano case things are less transparent. Performing the same considerations as above, but now with non-vanishing α_j in (5.192) yields for M not being Fano toric:

$$\boxed{\mathfrak{PO}(b) = \underbrace{\sum_{i=1}^{N} y_1^{v_{i_1}} \cdot \ldots \cdot y_n^{v_{i_n}} T^{l_i(p)}}_{=:\ \mathfrak{PO}_0(b)} + \sum_j c_j \cdot y_1^{v'_{j1}} \cdot \ldots \cdot y_n^{v'_{jn}} T^{l'_j(p) + \omega(\alpha_j)}} \tag{5.198}$$

Here we defined $c_j := \deg[ev : \mathcal{M}_1^{\mathrm{main}}(L(p), \alpha_j) \to L(p)]$ and

$$v'_j := \sum_{i=1}^{N} k_i^j v_i \quad , \quad l'_j(p) := \sum_{i=1}^{N} k_i^j l_i(p) \tag{5.199}$$

for appropriate $k_i^j \in \mathbb{Z}_{\geq 0}$ arising in the decompositions (5.192) for $\beta \in \pi_2(M, L(p))$.

5.6 Outlook for further studies: non-Fano toric manifolds

As a final announcement we remark that in most cases it is highly nontrivial to fully calculate the potential function and not just the leading order term $\mathfrak{PO}_0$.
The reader is referred to e.g. [FOOO3] where K. Fukaya et al. introduce among others the notion of bulk deformations in order to bypass these kind of difficulties. Further in [FOOO5] the authors explicitly calculate the full potential function for the non-Fano Hirzebruch surface (5.60) $H_{k=2}(\alpha)$.
The moment polytope

$$\begin{aligned} &\{(u_1, u_2) \in \mathbb{R}^2 \mid u_i \geq 0, u_2 \leq 1 - \alpha, u_1 + ku_2 \leq k\} = \\ =&\{u \mid l_i(u) = u_i \leq 0,\ l_3(u) = 1 - \alpha - u_2 \leq 0,\ l_4(u) = 2 - u_1 + 2u_2 \leq 0\} \end{aligned} \tag{5.200}$$

then yields a potential function of the form

$$\mathfrak{PO}(y; p) = \underbrace{y_1 T^{u_1} + y_2 T^{u_2} + y_1^{-1} y_2^{-2} T^{2-u_1-2u_2} + y_2^{-1} T^{1-\alpha-u_2}}_{\mathfrak{PO}_0(y_1, y_2; u_1, u_2)} + \underbrace{c}_{= T^{2\alpha}} T^{1-\alpha-u_2} y_2^{-1} . \tag{5.201}$$

Chapter 6

Applications: Properties of $\mathfrak{PO}$ - Relation to mathematics

6.1 Lagrangian Floer Cohomology for pairs L_0, L_1 of Lagrangian submanifolds

In order to face intersection issues for Lagrangian submanifolds we try to extend the ideas of chapter 5. Our aim is to define a Lagrangian Floer Cohomology theory, now not only for a single Lagrangian but for a pair L_0, L_1. Due to the work of K. Fukaya et al. this is possible at least if both intersect transversally or cleanly. We are still working in the toric setup, introduced in chapter 5, so we want both Lagrangians to arise as torus fibers over interior points in the corresponding moment polytope. The first section´s buildup is as follows:

Using K. Fukaya´s work [FOOO1] we extend the ideas about A_∞-algebras (chapter 3) and provide a purely algebraic description of A_∞-bimodule structures. We already clarified how filtered A_∞-algebras (C_i, m^i) are constructed out of *one* Lagrangian subtorus L_i. These preceding results come into play since we aim to discuss the notion of a left/right $(C_1, m^1)/(C_0, m^0)$ A_∞-bimodule. After this algebraic groundwork we pick these concepts and try to organize the setup of two Lagrangian submanifolds $L_0, L_1 \subset M$ in a (C_1, C_0) filtered A_∞-bimodule structure. This is done by incorporating classic Floer theoretic concepts namely the counting of stable broken trajectories connecting intersection points $p \in L_1 \cap L_0$.

As before after this challenging preparatory work, namely to put geometry in an algebraic dress, we are able to profit of its clear build up. Its defining A_∞-bimodule-relation is used to ask for possible coboundary operators necessary to define a Lagrangian Floer Cohomology theory. As the headline proposes we again can use the potential function $\mathfrak{PO}$ as a tool to check if possible candidates for coboundary operators actually square up to zero.

We even can further profit when properly analyzing the potential function $\mathfrak{PO}$. As presented in section 6.2 we use the result of proposition 4.6, namely that we are able to fully calculate $\mathfrak{PO}(x; p)$ for Fano toric manifolds, and discuss how its

derivatives can be used for concrete computations. It turns out that these can be used to derive an expression for the Lagrangian Floer coboundary operator δ_b. For general Fano toric manifolds M^{2n} we find at least one Lagrangian subtorus $L(p)$ (for $n = 2$ we even get a continuum of tori) for which we thereof can compute its Lagrangian Floer cohomology. Then relying on an important theorem of Fukaya et al. (Theorem J in [FOOO1]) this knowledge allows to give an lower estimate on the number of intersection points of $L(p)$ and $\psi(L(p))$, for $\psi \in Ham(M, \omega)$ being a Hamiltonian diffeomorphism, just depending on the Hofer norm $\|\psi\|$.
As in many parts of the text we are again following the ideas and the work of K. Fukaya et al. and rely to [FOOO1] and [FOOO2].

6.1.1 Algebraic Perception on A_∞-bimodules

Assume filtered A_∞-algebras (C_i, m^i) over $\Lambda_{0,nov}(R) \equiv \Lambda_{0,nov}$ (for the upcoming algebraic considerations a specification which ground ring R we are using is not necessary) are given for $i \in \{0, 1\}$. Picking up and extending the ideas of chapter 3 we declare degree $+1$ operators

$$n_{l_1,l_0} : B_{l_1}(C_1[1]) \otimes_{\Lambda_{0,nov}} D[1] \otimes_{\Lambda_{0,nov}} B_{l_0}(C_0[1]) \to D[1] \quad \text{for} \quad l_i \geq 0 \ . \tag{6.1}$$

Here $D[1]$ denotes the completion of a shifted, free, graded and filtered $\Lambda_{0,nov}$ module $\bigoplus_{m\in\mathbb{Z}} D^m$. As described in section 3.1.2, it is possible to complete here since

$$F^\lambda D^m := T^\lambda D^m \tag{6.2}$$

defines an energy filtration that is satisfying the properties of definition 3.3.
When additionally requiring the following filtration preserving property

$$\begin{aligned} n_{l_1,l_0}(F^{\lambda_1}(C_1[1])^{m_1} \otimes ... \otimes F^{\lambda_{l_1}}(C_1[1])^{m_{l_1}} \otimes F^{\lambda_0}(D[1])^{m_0} \otimes \\ \otimes F^{\lambda'_1}(C_0[1])^{m'_1} \otimes ... \otimes F^{\lambda'_{l_0}}(C_0[1])^{m'_{l_0}}) \subseteq F^{\sum_i \lambda_i + \lambda_0 + \sum_j \lambda'_j}(D[1])^{\sum_i m_i + m_0 + \sum_j m'_j + 1} \end{aligned} \tag{6.3}$$

we can further complete the domain of n_{l_1,l_0} and thus extend it to a homomorphism $\widehat{n}_{l_1,l_0}$ between the complete spaces (symbolized by a " $\widehat{\cdot}$ " sign)

$$\begin{aligned} \widehat{n}_{l_1,l_0} : (\bigoplus_{n,m} B_n(C_1[1]) \otimes_{\Lambda_{0,nov}} D[1] \otimes_{\Lambda_{0,nov}} B_m(C_0[1]))\widehat{\ } \longrightarrow \\ (\bigoplus_{\substack{n,m; \\ i+j=n+m-(l_0+l_1)}} B_i(C_1[1]) \otimes_{\Lambda_{0,nov}} D[1] \otimes_{\Lambda_{0,nov}} B_j(C_0[1]))\widehat{\ } =: BDB \ . \end{aligned} \tag{6.4}$$

Out of these we are then able to define a *bi-coderivation*

$$\widehat{d} : BDB \to BDB \ . \tag{6.5}$$

The operator has a comparably structure as the

$$\widehat{d}^{\,i} = \sum_k \widehat{m}^i_k : \widehat{B}(C_i[1]) \to \widehat{B}(C_i[1]) \tag{6.6}$$

defined from m^i in section 3.1.2. Precisely speaking it is defined as

$$\begin{aligned}
&\widehat{d}(y_{1,1}\otimes ... \otimes y_{1,l_1} \otimes x \otimes y_{0,1} \otimes ... \otimes y_{0,l_0}) := \\
&\quad\left(\sum_{i,j} (-1)^{\sum_{m=1}^{i} (\deg\ y_{1,m}+1)} y_{1,1} \otimes ... \otimes y_{1,i} \otimes \right. \\
&\qquad\left. \otimes\, n_{l_1-i,j}(y_{1,i+1} \otimes ... \otimes y_{1,l_1} \otimes x \otimes y_{0,1} \otimes ... \otimes y_{0,j}) \otimes y_{0,j+1} \otimes ... \otimes y_{0,l_0} \right) + \\
&\quad + \widehat{d}^{\,1}(y_{1,1} \otimes ... \otimes y_{1,l_1}) \otimes x \otimes y_{0,1} \otimes ... \otimes y_{0,l_0} + \\
&\quad + (-1)^{\sum_{m=1}^{l_1} (\deg\ y_{1,m}+1)+\deg\ x+1} y_{1,1} \otimes ... \otimes y_{1,l_1} \otimes x \otimes \widehat{d}^{\,0}(y_{0,1} \otimes ... \otimes y_{0,l_0}) \ .
\end{aligned} \tag{6.7}$$

Definition 6.1

(D, n) is called a (C_1, C_0) filtered A_∞-bimodule if $\widehat{d}$ is a coboundary operator for the complex BDB that is

$$\boxed{\widehat{d} \circ \widehat{d} = 0} \tag{6.8}$$

Further specifications for A_∞-bimodules are possible to declare:

(i) Being *unital filtered*, that is for units $\mathbf{e}_i$ of (C_i, m^i):

$$\begin{aligned} n_{1,0}(\mathbf{e}_1 \otimes x) = (-1)^{\deg\ y} n_{0,1}(x \otimes \mathbf{e}_0) = x \ ; \\ n_{l_1,l_0}(\cdots \otimes \mathbf{e}_i \otimes \cdots) = 0 \quad \text{for} \quad l_1 + l_0 \geq 2 \end{aligned} \tag{6.9}$$

(ii) Assume one has a free graded R module $\overline{D}$ and unfiltered A_∞-algebras $(\overline{C}_i, \overline{m}^i)$. One calls $(\overline{D}, \overline{n})$ a $(\overline{C}_1, \overline{C}_0)$ *unfiltered A_∞-bimodule* if analogously to above one gets

$$\widehat{\overline{d}} \circ \widehat{\overline{d}} = 0 \tag{6.10}$$

(iii) Comparably to the R-reduction described in section 3.1.2 and the way we construct A_∞-bimodules for concrete geometric setups later on, we can assume for our cases that

$$D \cong \overline{D} \otimes_R \Lambda_{0,nov} \tag{6.11}$$

and that

$$\overline{n}_{l_1,l_0} := n_{l_1,l_0} \mod \Lambda^+_{0,nov} \tag{6.12}$$

does not contain elements of $R[e, e^{-1}]$. The thereby arising $(\overline{C}_1, \overline{C}_0)$ unfiltered A_∞-bimodule $(\overline{D}, \overline{n})$ is called the *R-reduction* of (D, n). Here $(\overline{C}_i, \overline{m}^i)$ denotes the R-reduced unfiltered A_∞-algebra of (C_i, m^i).

(iv) (D, n) is called *G-gapped* if the homomorphisms n_{l_1,l_0} decompose as

$$n_{l_1,l_0} = \sum_i e^{n_i} T^{\lambda_i} n_{l_1,l_0,i} \tag{6.13}$$

for $(\lambda_i, n_i) \in G$ for $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ being a submonoid satisfying condition 3.1. When describing geometry and then assigning the symplectic volume to λ_i and the Maslov index to $2n_i$ we get that all the geometrically constructed A_∞-bimodules are G-gapped.

(v) Remark that there is a notion of filtered A_∞-*bimodule homomorphisms*

$$D \xrightarrow{\varphi} D' \ . \tag{6.14}$$

We do not discuss the details here since they are not needed in the following and refer the reader to section 3.7.2. of [FOOO1].

The precise way of defining these concepts is very similar to how we declared them in section 3.1.2. We refer to Definition 3.7.5 of [FOOO1] for an exact description.

6.1.2 From Geometry to A_∞-bimodules

As outlined in the motivation of this chapter, we first assume that the pair of connected Lagrangian submanifolds L_0, L_1 in (M, ω) intersects transversally.
We consider pairs (l, w) defined as follows:

- l is a path connecting L_0, L_1 that is

$$\{l \in C^0([0,1], M) \mid l(0) \in L_0, l(1) \in L_1\} =: \Omega(L_0, L_1) \ . \tag{6.15}$$

For a fixed base path l_0 we take $\Omega(L_0, L_1; l_0) \subset \Omega(L_0, L_1)$ as the connected component containing l_0.
Additionally we make the choice:

- $w \in C^0([0,1]^2, M)$ is a path in the path space satisfying the boundary conditions

$$\begin{aligned} &w(s,0) \in L_0,\ w(s,1) \in L_1 \quad \text{for all} \quad s \in [0,1] \\ &w(0,t) = l_0(t),\ w(1,t) = l(t) \quad \text{for all} \quad t \in [0,1] \ . \end{aligned} \tag{6.16}$$

The space $\widetilde{\Omega}(L_0, L_1; l_0)$ (often denoted as the *Novikov covering space*) contains equivalence classes $[l, w]$ whereas the equivalence relation $(l, w) \sim (l, w')$ is defined as follows:
For the concatenation $\overline{w}\#w'$ ($\overline{w}(s,t) := w(1-s,t)$ for $\overline{w}, w'$ as in (6.16)) we can calculate its symplectic energy via

$$I_\omega(\overline{w}\#w') := \int_{[0,1]^2} (\overline{w}\#w')^*\omega \ . \tag{6.17}$$

With

$$I_\mu(\overline{w}\#w') \tag{6.18}$$

we want to denote the Maslov index (see section 2.2) of the bundle pair

$$(\overline{w}\#w')^*TM \ , \ (\overline{w}\#w')_0^*TL_0 \sqcup (\overline{w}\#w')_1^*TL_1 \tag{6.19}$$

for $(\overline{w}\#w')_i(s) := (\overline{w}\#w')(s,i)$ $(i \in \{0,1\})$.
So we project pairs $(l,w),(l,w')$ with $l = l'$ and

$$I_\omega(\overline{w}\#w') = I_\mu(\overline{w}\#w') = 0 \tag{6.20}$$

into the equivalence class $[l,w]$. These shall form the elements of $\widetilde{\Omega}(L_0,L_1;l_0)$.
On this set one considers an *action functional*

$$\begin{aligned} \mathcal{A} : \widetilde{\Omega}(L_0,L_1;l_0) &\to \mathbb{R} \\ [l,w] &\mapsto \int w^*\omega \ . \end{aligned} \tag{6.21}$$

We refer the reader to section 2.3. of [FOOO1] where the well-definedness of this functional $\mathcal{A}$ is proven. The set of critical points of $\mathcal{A}$ is denoted by $Cr_{\mathcal{A}}(L_0,L_1;l_0)$. Due to how $\mathcal{A}$ is defined one deduces that this set consists of elements of the form $[l_p,w]$ for constant paths $l_p(t) \equiv p$ for $p \in L_0 \cap L_1$.
Our aim now is to define a free graded and filtered $\Lambda_{0,nov}$ module D out of these ingredients.
So first set $\widehat{DF}(L_0,L_1;\Lambda_{0,nov}) \equiv \widehat{DF}$ as the completion of

$$\Big(\bigoplus_{l_0} \bigoplus_{\substack{[l_p,w]\in \\ Cr_{\mathcal{A}}(L_0,L_1;l_0)}} \mathbb{Q}[l_p,w]\Big) \otimes_{\mathbb{Q}} \Lambda_{nov}^{R=\mathbb{Q}} \ . \tag{6.22}$$

A filtration $\mathcal{F}$ on $\widehat{DF}$ is declared by

$$F^{\lambda'}\widehat{DF}(L_0,L_1;\Lambda_{nov}) \tag{6.23}$$

containing elements with

$$\lambda + \int w^*\omega \geq \lambda' \tag{6.24}$$

for λ being the superscript of the formal parameter T. Recalling the meaning of the parameter λ for the construction of the A_∞-algebra in section 5.3, namely the characterization of the energy of the attached pseudo-holomorphic curves, we see that (6.24) can be interpreted as an energy bound from below.
We additionally define an equivalence relation on $\widehat{D}$ and denote

$$\widehat{DF}(L_0,L_1;\Lambda_{nov})/\sim \; =: DF(L_0,L_1;\Lambda_{nov}) \equiv DF \ . \tag{6.25}$$

Elements of the form $e^\mu T^\lambda[l_p,w], e^{\mu'}T^{\lambda'}[l_{p'},w']$ project to the equivalence class

$$\big[e^\mu T^\lambda[l_p,w]\big] \underbrace{\equiv}_{\substack{\text{abuse of} \\ \text{notation}}} e^\mu T^\lambda[l_p,w] \tag{6.26}$$

if the following equalities are fulfilled:

(i) $p = p'$

(ii) $\lambda + \int w^*\omega = \lambda' + \int w'^*\omega$

(iii) $2\mu + \mu([l_p, w]) = 2\mu' + \mu([l_{p'}, w'])$

The way how to define a Maslov index $\mu([l_p, w])$ is described in section 2.2.2. of [FOOO1] and can be summarized as defining it as the Maslov index of the bundle pair

$$(w^*TM, \lambda_w) \ . \tag{6.27}$$

Here the Lagrangian subbundle $\lambda_w \to \partial[0,1]$ is defined as

$$\begin{aligned} &\lambda_w(s,0) = T_{w(s,0)}L_0; \quad \lambda_w(0,t) = \text{section } \lambda^0 \text{ in } l_0^*(\bigcup_p \{\text{oriented Lagr. in } T_pM\}) \\ &\lambda_w(s,1) = T_{w(s,1)}L_1; \quad \lambda_w(1,t) = \text{path } \alpha^p \text{ in } \{\text{oriented Lagr. in } T_pM\} \ . \end{aligned} \tag{6.28}$$

It is possible to show that $\mu([l_p, w])$ does not depend on the chosen α^p. The reference section λ_0 is fixed once in order to provide an absolute grading for the Floer complex. We do not this fact for following discussion and since the chapter´s purpose is to provide a basic impression for A_∞-bimodules describing geometry, the interested reader is referred to chapter 5.1. of [FOOO1].

Let us return to the definition of the equivalence relation $\sim$. Point (ii) is necessary to assign the described filtration $\mathcal{F}$ to DF. The searched $\Lambda_{0,nov}$ module D is defined as the submodule with non-negative total energy that is

$$F^0 DF(L_0, L_1; \Lambda_{nov}) =: D(L_0, L_1; \Lambda_{0,nov}) \equiv D \ . \tag{6.29}$$

Point (iii) is used to declare a grading given by

$$2\mu + \mu([l_p, w]) \tag{6.30}$$

for elements in D. For the filtration we take $F^{\lambda'}D$ containing elements with exponents $\geq \lambda'$ of the generator T.
As proclaimed in the algebraic part of this section, D can be seen as a complete, free graded and filtered $\Lambda_{0,nov}$ module.

Using the ideas of section 5.3 namely taking $(H^*(L_i, \Lambda^{\mathbb{R}}_{0,nov}), m^i)$ as the underlying A_∞-algebras allows us to define the operators n_{l_0,l_1} as a mapping

$$\begin{aligned} n_{l_1,l_0} : H^{r_1}(L_1, \Lambda^{\mathbb{R}}_{0,nov}) \otimes ... \otimes H^{r_{l_1}}(L_1, \Lambda^{\mathbb{R}}_{0,nov}) \otimes D(L_0, L_1; \Lambda_{0,nov}) \otimes \\ \otimes H^{r'_1}(L_0, \Lambda^{\mathbb{R}}_{0,nov}) \otimes ... \otimes H^{r'_{l_0}}(L_0, \Lambda^{\mathbb{R}}_{0,nov}) \longrightarrow D(L_0, L_1; \Lambda_{0,nov}) \end{aligned} \tag{6.31}$$

defined via

$$\begin{aligned} e^{\mu_1^{(1)}} T^{\lambda_1^{(1)}} \alpha_1^{(1)} \otimes ... \otimes e^{\mu_{l_1}^{(1)}} T^{\lambda_{l_1}^{(1)}} \alpha_{l_1}^{(1)} \otimes e^{\mu} T^{\lambda} [l_p, w] \otimes e^{\mu_1^{(0)}} T^{\lambda_1^{(0)}} \alpha_1^{(0)} \otimes ... \otimes e^{\mu_{l_0}^{(0)}} T^{\lambda_{l_0}^{(0)}} \alpha_{l_0}^{(0)} \\ \longmapsto \\ \sum_{\substack{[l_p,w]\in \\ Cr_{\mathcal{A}}(L_0,L_1;l_0)}} \#(\underbrace{\mathcal{M}_{k_1,k_0}(L_0, L_1; [l_p, w], [l_q, w']; \vec{\alpha}^{(0)}, \vec{\alpha}^{(1)})^{\mathfrak{s}}}_{\mathcal{M}_{k_1,k_0}(\cdots)}) \ e^{\mu'} T^{\lambda'} [l_q, w'] \end{aligned} \tag{6.32}$$

for $\lambda' = \sum_{i=1}^{l_1} \lambda_i^{(1)} + \lambda + \sum_{i=1}^{l_0} \lambda_i^{(0)}$ and $\mu' = \sum_{i=1}^{l_1} \mu_i^{(1)} + \mu + \sum_{i=1}^{l_0} \mu_i^{(0)}$.
For an exact definition of the appearing moduli space $\mathcal{M}_{k_1,k_0}(\cdots)$ we refer to Proposition 3.7.26. of [FOOO1], where the authors discuss important issues namely:

- $\mathcal{M}_{l_1,l_0}(\cdots)$ is compact
- ev is weakly submersive, strongly continuous, smooth
- $\mathcal{M}_{l_1,l_0}(\cdots)$ can be equipped with an oriented Kuranishi structure

In Proposition 3.7.36. they further derived a formula to calculate its virtual dimension and showed that it is possible to find a transversal compatible multisection in order to perturb $\mathcal{M} \to \mathcal{M}^{\mathfrak{s}}$. Since compactness is guaranteed we can actually count the number of (weighted) points of the 'zero' dimensional moduli spaces.

For our purpose it is enough to think of $\mathcal{M}_{l_1,l_0}(L_0, L_1; [l_p, w], [l_q, w'])$ as the moduli space of marked stable broken pseudo-holomorphic curves (Fig. 6.1)

$$u : \mathbb{R} \times [0, 1] \to M \tag{6.33}$$

satisfying certain boundary conditions described in chapter 3.7. of [FOOO1].
The additional specification $\mathcal{M}_{l_1,l_0}(...; \vec{\alpha}^{(0)}, \vec{\alpha}^{(1)})$ means that we only regard curves whose images of the marked points attach

$$a_j^{(i)} = \mathrm{PD}(\alpha_j^{(i)}) \in H_{n-\deg\ \alpha_j^{(i)}}(L_i, \mathbb{R}). \tag{6.34}$$

Remark that requiring such constraints reduces the dimension of $\mathcal{M}$ by

$$\sum_{j=1}^{k_0} \deg\ a_j^{(0)} + \sum_{j=1}^{k_1} \deg\ a_j^{(0)}. \tag{6.35}$$

Adopting Theorem 3.7.21. of [FOOO1] we get:

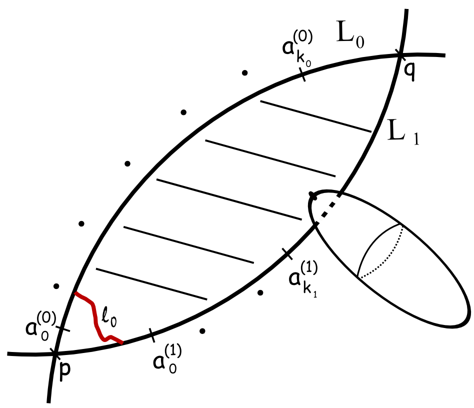


Figure 6.1: Visualizing $\mathcal{M}_{k_1,k_0}(L_0, L_1; [l_p, w], [l_q, w']; \vec{\alpha}^{(0)}, \vec{\alpha}^{(1)})$

Theorem 6.1

The constructed $(D(L_0, L_1; \Lambda^{\mathbb{R}}_{0,nov}), n)$ is a right $(H^*(L_0, \Lambda^{\mathbb{R}}_{0,nov}), m^0)$ and left $(H^*(L_1, \Lambda^{\mathbb{R}}_{0,nov}), m^1)$ unital filtered A_∞-bimodule with units $\mathbf{e}_i = PD([L_i])$.

Remark 6.1. *We finally remark that the requirement of L_0, L_1 being transversal can be replaced by requiring that the Lagrangians intersect cleanly. See Definition 3.7.48. of [FOOO1] for the meaning of clean intersections, here we just remark that the case of*

$$L_0 = L_1 \equiv L \tag{6.36}$$

is covered by this definition.
In Theorem 3.7.72. of [FOOO1] Fukaya et al. showed that under these assumption an A_∞-bimodule structure can be built (similarly as above by counting marked stable broken Floer trajectories).
Concerning later considerations, where we present a method to actually compute the Lagrangian Floer Cohomology (see section 6.2.1), we highlight that the setup of one Lagrangian $L \subset M$ can either be put in the algebraic language of A_∞-algebras (see section 5.3) or as remarked here into notion of A_∞-bimodules. We pick up this aspect again in Remark 6.2 (ii) where we discuss how these structures are related and, even more important for our intention, easily see that the two thereof arising Lagrangian Floer Cohomologies are actually identical.

6.1.3 From A_∞-bimodules to a Lagrangian Floer Cohomology

As in section 3.2.1 and 3.2.2 we are able to follow two different approaches in order to define a Lagrangian Floer Cohomology theory. Tying up the situation of theorem 6.1, we therefore try to declare a coboundary operator out of the given left, right

$$(H^*(L_1, \Lambda^{\mathbb{R}}_{0,nov}), m^1) \quad , \quad (H^*(L_0, \Lambda^{\mathbb{R}}_{0,nov}), m^0) \tag{6.37}$$

filtered A_∞-bimodule $(D(L_0, L_1; \Lambda^{\mathbb{R}}_{0,nov}), n)$. As in section 3.2 this way of approaching is only possible if the underlying A_∞-algebras are at least weakly unobstructed (see Definition 3.8).
The "strict" approach again asks for possible deformations of the maps $\{n_{l_1,l_0}\}_{l_1,l_0 \geq 0}$ by using strict Maurer-Cartan solutions, whereas in the weak case the potential functions $\mathfrak{PO}_i$ are used to check whether candidates for possible coboundary operators really square up to zero.
Remark that the upcoming concepts work for general A_∞-bimodules not just the one of theorem 6.1. Since we already presented the transfer of formulating the "$L_0, L_1 \subset M$" setup in an algebraic A_∞-bimodule fashion, we work with these constructed data. This helps to keep the geometry in mind when doing the algebra.

<u>Strict unobstructedness via deformations</u>:

We want to make use of strict (meaning $\widehat{d}^i(e^{b_i}) = 0$) Maurer-Cartan solutions (bounding cochains)

$$b_i \in H^1(L_i, \Lambda^{\mathbb{R}}_{0,nov}) \tag{6.38}$$

(if existing!) for $i \in \{0,1\}$.
For general

$$b_i \in H^1(L_i, \Lambda_{0,nov}^{\mathbb{R}}) \quad \text{with} \quad b_i \equiv 0 \bmod \Lambda_{0,nov}^{\mathbb{R}} \tag{6.39}$$

one deforms the given A_∞-bimodule structure via first replacing the A_∞-algebra homomorphisms $\{m_l^0\}_{l\geq 0}$ and $\{m_l^1\}_{l\geq 0}$ by

$$m^i = \{m_l^i\} \to \{m_l^{i,b_i}\} = m^{i,b_i} \tag{6.40}$$

as described in section 3.2.1. Then one further performs a replacement of the A_∞-bimodule homomorphisms $\{n_{l_1,l_0}\}_{l_1,l_0\geq 0}$ via

$$\begin{aligned} & n_{l_1,l_0}(y_{1,1},...,y_{1,l_1},x,y_{0,1},...,y_{0,l_0}) \to {}^{b_1}n^{b_0}_{l_1,l_0}(y_{1,1},...,y_{1,l_1},x,y_{0,1},...,y_{0,l_0}) := \\ &= \sum_{\substack{k_0,...,k_{l_1} \\ k'_0,...,k'_{l_0}}} n_{l_1+\sum k_i, l_0+\sum k'_i}(\underbrace{b_1,...,b_1}_{k_0}, y_{1,1}, \underbrace{b_1,...,b_1}_{k_1}, ..., \underbrace{b_1,...,b_1}_{k_{l_1}-1}, y_{1,l_1}, \underbrace{b_1,...,b_1}_{k_{l_1}}, x, \\ &\qquad \underbrace{b_0,...,b_0}_{k'_0}, y_{0,1}, \underbrace{b_0,...,b_0}_{k'_1}, ..., \underbrace{b_0,...,b_0}_{k'_{l_0}-1}, y_{0,l_0}, \underbrace{b_0,...,b_0}_{k'_{l_0}}) \equiv \\ &\equiv n(e^{b_1}y_{1,1}e^{b_1},...,e^{b_1}y_{1,l_1}e^{b_1},x,e^{b_0}y_{0,1}e^{b_0},...,e^{b_0}y_{0,l_0}e^{b_0}) \ . \end{aligned} \tag{6.41}$$

Remark that the maps ${}^{b_1}n^{b_0}, m^{0,b_0}, m^{1,b_1}$ (as in (6.7)) induce an operator

$${}^{b_1}\widehat{d}^{\,b_0} : BDB \to BDB \tag{6.42}$$

via

$$\begin{aligned} & {}^{b_1}\widehat{d}^{\,b_0}(\underline{y}_1 \otimes x \otimes \underline{y}_0) := \\ &\quad = \sum (-1)^{\cdots} y_{1,1} \otimes ... \otimes y_{1,i} \otimes {}^{b_1}n^{b_0}_{l_1-i,j}(\cdots \otimes x \otimes \cdots) \otimes y_{0,j+1} \otimes ... \otimes y_{0,l_0} + \\ &\quad + \widehat{d}^{\,1,b_1}\underline{y}_1 \otimes x \otimes \underline{y}_0 + (-1)^{\cdots}\underline{y}_1 \otimes x \otimes \widehat{d}^{\,0,b_0}\underline{y}_0 \ . \end{aligned} \tag{6.43}$$

As in the proof of proposition 3.1 by directly comparing

$$0 = (\widehat{d} \circ \widehat{d})(e^{b_1}y_{1,1}e^{b_1} \otimes ... \otimes e^{b_1}y_{1,l_1}e^{b_1} \otimes x \otimes e^{b_0}y_{0,1}e^{b_0} \otimes ... \otimes e^{b_0}y_{0,l_0}e^{b_0}) \tag{6.44}$$

with

$$({}^{b_1}\widehat{d}^{\,b_0} \circ {}^{b_1}\widehat{d}^{\,b_0})(y_{1,1} \otimes ... \otimes y_{1,l_1} \otimes x \otimes y_{0,1} \otimes ... \otimes y_{0,l_0}) \tag{6.45}$$

yields

$$\boxed{{}^{b_1}\widehat{d}^{\,b_0} \circ {}^{b_1}\widehat{d}^{\,b_0} = 0 \ .} \tag{6.46}$$

This means that $(D(L_0,L_1;\Lambda_{0,nov}), \ {}^{b_1}n^{b_0})$ is again a

$$(H^*(L_1, \Lambda_{0,nov}^{\mathbb{R}}), m^{1,b_1}) \quad , \quad (H^*(L_0, \Lambda_{0,nov}^{\mathbb{R}}), m^{0,b_0}) \tag{6.47}$$

filtered A_∞-bimodule.

When additionally requiring

$$b_i \in \mathcal{M}_{\text{strict}}(H^*(L_i, \Lambda^{\mathbb{R}}_{0,nov})), \tag{6.48}$$

that is b_0, b_1 being strict Maurer-Cartan solutions the A_∞-bimodule defining equation, (6.7) boils down to

$$\begin{aligned} 0 &= (\widehat{d} \circ \widehat{d})(e^{b_1} x e^{b_0}) \overbrace{=}^{\deg b+1=2} \\ &= \widehat{d}\Big(e^{b_1} n(e^{b_1} x e^{b_0}) e^{b_0} + \widehat{d}^1(e^{b_1}) \otimes x \otimes e^{b_0} + (-1)^{\deg x+1} e^{b_1} \otimes x \otimes \widehat{d}^1(e^{b_0}) \Big) \underbrace{=}_{\widehat{d}^i(e^{b_i})=0} \\ &= n(e^{b_1} n(e^{b_1} x e^{b_0}) e^{b_0}) \ . \end{aligned} \tag{6.49}$$

In summary we get that the degree +1 operator

$$\delta_{b_1,b_0}(x) := n(e^{b_1} x e^{b_0}) \tag{6.50}$$

can be seen as a coboundary operator ($\delta_{b_1,b_0} \circ \delta_{b_1,b_0} = 0$)

$$\delta_{b_1,b_0} : D[1] \to D[1] \ . \tag{6.51}$$

Weak unobstructedness via comparison of $\mathfrak{PO}_0$ and $\mathfrak{PO}_1$:

Comparably as in section 3.2.2 we can however follow a bit less restrictive approach for declaring a coboundary operation. This is done by considering a weakened version of $\widehat{d}^i(e^{b_i}) = 0$, namely we search weak Maurer-Cartan solutions, that is elements

$$b_i \in H^1(L_i, \Lambda^{\mathbb{R}}_{0,nov}) \tag{6.52}$$

fulfilling

$$m^i(e^{b_i}) = c_i \cdot e \cdot \mathbf{e}_i \quad \text{for} \quad c_i \in \Lambda^{+(0)}_{0,nov}(\mathbb{R}) \ . \tag{6.53}$$

Since $\mathbf{e}_i$ ($i = 0, 1$) denote the units of the underlying A_∞-algebras, this usage of elements of

$$\mathcal{M}_{\text{weak}}(C) \tag{6.54}$$

is in general, for not further specified (C_1, C_0) A_∞-bimodules, only possible if they are considered to be unital. We are describing geometry in an A_∞-bimodule fashion and due to Theorem 6.1, stating that PD$[L_i]$ serve as units, this weak approach is justified for our purpose.
Recall (3.126) namely that these weak Maurer-Cartan solutions form the domain of the potential functions

$$\begin{aligned} \mathfrak{PO}_i : \mathcal{M}_{\text{weak}}(H^*(L_i, \Lambda^{\mathbb{R}}_{0,nov})) &\to \Lambda^{+(0)}_{0,nov}(\mathbb{R}) \\ b_i &\mapsto c_i \end{aligned} \tag{6.55}$$

defined by

$$m^i(e^{b_i}) = \mathfrak{PO}_i(b_i) \cdot e \cdot \mathbf{e}_i \ . \tag{6.56}$$

Declaring an operator

$$\begin{aligned} \delta_{b_1,b_0} : D(L_0, L_1; \Lambda^{\mathbb{R}}_{0,nov}) &\to D(L_0, L_1; \Lambda^{\mathbb{R}}_{0,nov}) \\ x &\mapsto \sum_{k_1,k_0} n_{k_1,k_0}(\underbrace{b_1, ..., b_1}_{k_0}, x, \underbrace{b_0, ..., b_0}_{k_0}) \\ &\equiv n(e^{b_1} x e^{b_0}) \end{aligned} \tag{6.57}$$

similarly defined as in (6.50) but now for

$$b_i \in \mathcal{M}_{\text{weak}}(H^*(L_i, \Lambda^{\mathbb{R}}_{0,nov})) \tag{6.58}$$

we try to derive an equation that links δ_{b_1,b_0} with $\mathfrak{PO}_i$.
As the headline of this chapter states we aim to describe applications of the potential functions. Here a comparison of $\mathfrak{PO}_0, \mathfrak{PO}_1$ can be used in order to check if δ_{b_1,b_0} serves as a coboundary operator.

According to the definition of the bi-coderivation

$$\begin{aligned} \widehat{d} : \widehat{B}(C_1[1])\widehat{\otimes}_{\Lambda^{\mathbb{R}}_{0,nov}} D[1]\widehat{\otimes}_{\Lambda^{\mathbb{R}}_{0,nov}} \widehat{B}(C_0[1]) \to& \widehat{B}(C_1[1])\widehat{\otimes}_{\Lambda^{\mathbb{R}}_{0,nov}} D[1]\widehat{\otimes}_{\Lambda^{\mathbb{R}}_{0,nov}} \widehat{B}(C_0[1]) \\ \underline{y}_1 \otimes x \otimes \underline{y}_0 \mapsto& \sum_{i,j} (-1)^{\cdots} y_{1,1} \otimes ... \otimes y_{1,i} \otimes n_{l_1-i,j}(\cdots \\ & \cdots \otimes x \otimes \cdots) \otimes y_{0,j+1} \otimes ... \otimes y_{0,l_0} + \\ & + \widehat{d}^1 \underline{y}_1 \otimes x \otimes \underline{y}_0 + (-1)^{\cdots} \underline{y}_1 \otimes x \otimes \widehat{d}^0 \underline{y}_0 \end{aligned} \tag{6.59}$$

we get that the image of $\widehat{d} \circ \widehat{d}$ in the component

$$\widehat{B}_0(C_1[1])\widehat{\otimes}_{\Lambda^{\mathbb{R}}_{0,nov}} D[1]\widehat{\otimes}_{\Lambda^{\mathbb{R}}_{0,nov}} \widehat{B}_0(C_0[1]) \tag{6.60}$$

is given by $n \circ \widehat{d}$ (recall $\widehat{d^i} = \sum\limits_{k=0}^{\infty} \widehat{m}^i_k$ with $\widehat{m}^i_k = 0$ for $k > n$). Using the required A_∞-bimodule relation (6.8) we thus deduce

$$\widehat{d} \circ \widehat{d} = 0 \Rightarrow n \circ \widehat{d} = 0 \ . \tag{6.61}$$

For $x \in D(L_0, L_1; \Lambda^{\mathbb{R}}_{0,nov})$ and the fact

$$(-1)^{\deg\ b_i + 1} = (-1)^2 = 1 \tag{6.62}$$

we continue as follows:

$$
\begin{aligned}
0 &= n(\widehat{d}(e^{b_1}xe^{b_0})) \overbrace{=}^{(6.59)} \\
&= n(e^{b_1}n(e^{b_1}xe^{b_0})e^{b_0}) + n(\widehat{d}^1(e^{b_1})xe^{b_0}) + (-1)^{\deg\, x+1}n(e^{b_1}x\widehat{d}^0(e^{b_0})) \overbrace{=}^{\delta_{b_1,b_0}(\cdot):=n(e^{b_1}\cdot e^{b_0})} \\
&= (\delta_{b_1,b_0}\circ\delta_{b_1,b_0})(x) + n(\widehat{d}^1(e^{b_1})xe^{b_0}) - (-1)^{\deg\, x}n(e^{b_1}x\widehat{d}^0(e^{b_0})) \overbrace{=}^{(3.102)} \\
&= (\delta_{b_1,b_0}\circ\delta_{b_1,b_0})(x) + n(e^{b_1}m^1(e^{b_1})e^{b_1}xe^{b_0}) - (-1)^{\deg\, x}n(e^{b_1}xe^{b_0}m^0(e^{b_0})e^{b_0}) \overbrace{=}^{(6.56)} \\
&= (\delta_{b_1,b_0}\circ\delta_{b_1,b_0})(x) + n(e^{b_1}\mathfrak{PO}_1(b_1)\cdot e\cdot \mathbf{e}_1 e^{b_1}xe^{b_0}) + \\
&\qquad - (-1)^{\deg\, x}n(e^{b_1}xe^{b_0}\mathfrak{PO}_0(b_0)\cdot e\cdot \mathbf{e}_0 e^{b_0}) \overbrace{=}^{(6.9)} \\
&= (\delta_{b_1,b_0}\circ\delta_{b_1,b_0})(x) + \mathfrak{PO}_1(b_1)\cdot e\cdot x - (-1)^{2\deg\, x}\,\mathfrak{PO}_0(b_0)\cdot e\cdot x
\end{aligned}
\tag{6.63}
$$

As proclaimed the potential function can be used to check the property if δ_{b_1,b_0} serves as a coboundary operator for $D(L_0, L_1; \Lambda^{\mathbb{R}}_{0,nov})$:

$$\boxed{\delta_{b_1,b_0}(\delta_{b_1,b_0}(x)) = (\mathfrak{PO}_0(b_0) - \mathfrak{PO}_1(b_1))\cdot e\cdot x} \tag{6.64}$$

Defining the Lagrangian Floer Cohomology:

Definition 6.2

For $b_i \in H^1(L_i, \Lambda^{\mathbb{R}}_{0,nov})$ with $b_i \equiv 0$ mod $\Lambda^{\mathbb{R}}_{0,nov}$ ($i \in \{0,1\}$) satisfying either the strict

- $$\mathfrak{PO}_i(b_i) = 0 \tag{6.65}$$

or the weak condition

- $$\mathfrak{PO}_0(b_0) = \mathfrak{PO}_1(b_1). \tag{6.66}$$

the Lagrangian Floer Cohomology for the left/right $(H^*(L_1, \Lambda^{\mathbb{R}}_{0,nov}), H^*(L_0, \Lambda^{\mathbb{R}}_{0,nov}))$ filtered unital A_∞-bimodule $(D(L_0, L_1; \Lambda^{\mathbb{R}}_{0,nov}), n)$ is defined by

$$HF((L_1, b_1), (L_0, b_0); \Lambda^{\mathbb{R}}_{0,nov}) := Ker\ \delta_{b_1,b_0}/Im\ \delta_{b_1,b_0}\ . \tag{6.67}$$

Remark 6.2. *(i) As already remarked at the beginning of this section 6.1.3 a Lagrangian Floer Cohomology can in general be defined for (unital) filtered A_∞-bimodules as long as the underlying A_∞-algebras are (weakly) unobstructed, that is*

$$\mathcal{M}_{strict/weak}(H^*(L_i, \Lambda^{\mathbb{R}}_{0,nov})) \neq \emptyset. \tag{6.68}$$

The preceding definition is formulated more specific for not drifting away from our algebraic view on geometry constructed in section 6.1.2.

(ii) We are picking up the ideas brought up in Remark 6.1. For the most trivial case of only one Lagrangian submanifold

$$L_0 = L_1 = L \qquad \textit{and thus} \qquad m^0 = m^1 = m\ . \tag{6.69}$$

the described theory about A_∞-algebras (see chapter 3) can be boiled up to an A_∞-bimodule description (remark that L and L intersect cleanly).
Via setting

$$n_{l_1,l_0} := m_{l_1+l_0+1} \tag{6.70}$$

the A_∞-algebra $(H^*(L,\Lambda^{\mathbb{R}}_{0,nov}), m)$ extends to an left $(H^*(L,\Lambda^{\mathbb{R}}_{0,nov}), m)$ and right $(H^*(L,\Lambda^{\mathbb{R}}_{0,nov}), m)$ A_∞-bimodule $(H^*(L,\Lambda^{\mathbb{R}}_{0,nov}), m)$. Due to Proposition 3.7.75. of [FOOO1] this A_∞-bimodule coincides with the one arising from the general construction for cleanly intersecting Lagrangian submanifolds outlined in remark 6.1. This in turn implies

$$\begin{aligned}\widehat{d}(\underline{y}_1 \otimes x \otimes \underline{y}_0) = &\sum_{i,j}(-1)^{\cdots} y_{1,1}\otimes ... \otimes y_{1,i} \otimes m_{l_1-i+j+1}(\cdots \otimes x \otimes \cdots)\otimes \\ &\qquad \otimes y_{0,j+1}\otimes ... \otimes y_{0,l_0} + \widehat{d}^1\underline{y}_1 \otimes x \otimes \underline{y}_0 + (-1)^{\cdots}\underline{y}_1 \otimes x \otimes \widehat{d}^0\underline{y}_0 \\ \equiv &\widehat{d}^0(\underline{y}_1 \otimes x \otimes \underline{y}_0) = \widehat{d}^1(\underline{y}_1 \otimes x \otimes \underline{y}_0)\end{aligned} \tag{6.71}$$

for

$$y_{1,k}, y_{0,l}, x \in H^*(L,\Lambda^{\mathbb{R}}_{0,nov}) \quad \textit{with} \quad k,l > 0\ . \tag{6.72}$$

Additionally we can use

$$PD[L] = \boldsymbol{e}_0 = \boldsymbol{e}_1 = \boldsymbol{e} \tag{6.73}$$

as A_∞-algebra/A_∞-bimodule units and if $(H^*(L,\Lambda^{\mathbb{R}}_{0,nov}), m)$ is (strictly/weakly) unobstructed a (strict/weak) Maurer-Cartan solution

$$b_0 = b_1 \equiv b \tag{6.74}$$

that serves to define a coboundary operator

$$\delta_{b_0,b_1}(\cdot) = \underbrace{\delta_{b,b}(\cdot)}_{\textit{see } (3.108)} = \underbrace{\delta_b(\cdot)}_{\textit{see } (3.119)} = m(e^b \cdot e^b) \tag{6.75}$$

on $H^*(L,\Lambda^{\mathbb{R}}_{0,nov})$ out of the A_∞-algebra $(H^*(L,\Lambda^{\mathbb{R}}_{0,nov}), m)$.
With the help of the trivial identity

$$\mathfrak{PO}_0(b) = \mathfrak{PO}_1(b) = \mathfrak{PO}(b) \tag{6.76}$$

and due to (6.64) we know that

$$\delta'_{b,b}(\cdot) := n(e^b \cdot e^b) \tag{6.77}$$

is a coboundary on $H^*(L,\Lambda^{\mathbb{R}}_{0,nov})$ defined out of the A_∞-bimodule $(H^*(L,\Lambda^{\mathbb{R}}_{0,nov}),m)$. Further

$$\begin{aligned} m(e^b\cdot e^b) &= \sum_{l_1,l_0\geq 0} m_{l_1+l_0+1}(\underbrace{b,...,b}_{l_1},x,\underbrace{b,...,b}_{l_0}) = \\ &= \sum_{l_1,l_0\geq 0} n_{l_1,l_0}(\underbrace{b,...,b}_{l_1},x,\underbrace{b,...,b}_{l_0}) = \\ &= n(e^b\cdot e^b) \end{aligned} \tag{6.78}$$

shows that the coboundary operators $\delta_{b,b},\delta'_{b,b}$ defined on $H^*(L,\Lambda^{\mathbb{R}}_{0,nov})$ and thus the Lagrangian Floer Cohomology

$$\underbrace{HF(L,b;\Lambda^{\mathbb{R}}_{0,nov})}_{see\ (5.127)} \quad and \quad \underbrace{HF((L,b),(L,b);\Lambda^{\mathbb{R}}_{0,nov})}_{see\ Def.\ 6.2} \tag{6.79}$$

coincide.

6.2 Examination of derivatives of $\mathfrak{PO}$

The motivation for the second section of this chapter, which tries to describe how to draw out possible applications of the beforehand developed concepts, is to work out how derivatives of the potential function $\mathfrak{PO}$ can be used to compute the Lagrangian Floer Cohomology.
With this knowledge in mind we are then well prepared to handle (non-)displaceability questions for Lagrangian submanifolds.
Remark that by abuse of notation that we write Λ_0, Λ_+ for

$$\Lambda_0(\mathbb{C})\ ,\ \Lambda_0^+(\mathbb{C}) \tag{6.80}$$

respectively.
Recall that for a compact, orientable n dimensional Lagrangian subtori

$$T^n \cong L(p) = \mu^{-1}(p) \quad \text{for} \quad p\in\mathring{\Delta} \tag{6.81}$$

in a compact toric $2n$ dimensional manifold M^{2n} (always assumed to be Fano in the following), due to the embedding

$$H^1(L(p);\Lambda_+) \hookrightarrow \mathcal{M}_{\text{weak}}(L(p)) \quad \text{for} \quad p\in\mathring{\Delta} \tag{6.82}$$

derived in Proposition 5.5, the abstractly introduced potential function $\mathfrak{PO}$ (see Definition 3.126) can be restricted and then be written as

$$\mathfrak{PO} : H^1(L(p);\Lambda_+) \to \Lambda_+\ . \tag{6.83}$$

By using $\{e_1,...,e_n\}$ as an integral basis for $H^1(L(p);\mathbb{Z})\cong\mathbb{Z}^n$ (recall that we take it as the dual basis to the given integral basis $\{e_1^*,...,e_n^*\}$ of the lattice $N\cong\mathbb{Z}^n$

determining the underlying fan, Σ in $N \otimes \mathbb{R}$, of the toric manifold M), we further have

$$\bigcup_{p \in \mathring{\Delta}} H^1(L(p); \Lambda_+) \cong (\Lambda_+)^n \times (\mathring{\Delta}) \ . \tag{6.84}$$

Hence we get a coordinate description

$$\underbrace{\mathfrak{PO}(b = \sum_{i=1}^{n} x_i e_i)}_{\in \ H^1(L(p);\Lambda_+)} = \mathfrak{PO}(x;p) = \mathfrak{PO}(\underbrace{x_1, ..., x_n}_{\in (\Lambda_+)^n}; \underbrace{p_1, ..., p_n}_{p \in \mathring{\Delta}}) \ . \tag{6.85}$$

For $p \in \mathring{\Delta}$ fixed we regard $\mathfrak{PO}^p(x) := \mathfrak{PO}(x;p)$ as a function depending on $x = (x_1, ..., x_n)$.

6.2.1 Computation of $HF((L(p), b), (L(p), b); \Lambda_0)$

Adopting the language and especially the essence of Theorem 4.10. of [FOOO2], whose proof can be found in section 13 of the same article, we have the following:

Def./Prop. 6.1

A n dimensional Lagrangian torus fiber $L(p_0) \cong T^n$ ($p_0 \in \mathring{\Delta}$) in a $2n$ dimensional compact Fano toric manifold M^{2n} is called balanced if

$$HF^*((L(p_0), \rho, b), (L(p_0), \rho, b); \Lambda_0) \cong H^*(T^n; \Lambda_0) \ . \tag{6.86}$$

There is at least one such balanced Lagrangian torus fiber in M.
Especially in the case $n = 2$ we have a complete description of the Lagrangian Floer coboundary operator

$$m_1^{b_0} = \delta_{b_0} : H^*(L(p); \Lambda_0) \to H^*(L(p); \Lambda_0) \tag{6.87}$$

in terms of

$$\frac{\partial \mathfrak{PO}^p}{\partial x_i}|_{b=b_0} \quad \textit{for} \quad i \in \{1, 2\} \tag{6.88}$$

and p contained in a connected subset $I_{\mathfrak{PO}} \subset \mathring{\Delta}$ whose form is depending on $\mathfrak{PO}$. Precisely speaking for bases

$$\begin{aligned} \{\mathbf{e}_0\} \quad &\textit{of} \quad H^0(L(p), \mathbb{Z}) \cong \mathbb{Z} \\ \{\mathbf{e}_1, \mathbf{e}_2\} \quad &\textit{of} \quad H^1(L(p), \mathbb{Z}) \cong \mathbb{Z}^2 \\ \{\mathbf{e}_{12} = \mathbf{e}_1 \cup \mathbf{e}_2\} \quad &\textit{of} \quad H^2(L(p), \mathbb{Z}) \cong \mathbb{Z} \end{aligned} \tag{6.89}$$

we have

$$\begin{aligned} \delta_{b_0}(\mathbf{e}_0) &= 0 \\ \delta_{b_0}(\mathbf{e}_i) &= \frac{\partial \mathfrak{PO}^p}{\partial x_i}|_{b=b_0} \mathbf{e}_0 \\ \delta_{b_0}(\mathbf{e}_{12}) &= \frac{\partial \mathfrak{PO}^p}{\partial x_1}|_{b=b_0} \mathbf{e}_2 - \frac{\partial \mathfrak{PO}^p}{\partial x_2}|_{b=b_0} \mathbf{e}_1 \ . \end{aligned} \tag{6.90}$$

Remark 6.3. *We in particular highlight the $n = 2$ case here since, in contrast to the former general method of locating at least one balanced torus fiber $L(p_0)$, we even get continuum of 2 dimensional Lagrangian tori $L(p)$ for which we actually can compute its Lagrangian Floer Cohomology.*
Both phenomena are related in a way such that (6.90) holds for arbitrarily n, precisely speaking we have

$$\begin{aligned}
\delta_{b_0}(\boldsymbol{e}_0) &= 0 \\
\delta_{b_0}(\boldsymbol{e}_i) &= \frac{\partial \mathfrak{PO}^p}{\partial x_i}|_{b=b_0}\boldsymbol{e}_0 \\
\delta_{b_0}(\boldsymbol{e}_i \cup \boldsymbol{e}_j) &= \frac{\partial \mathfrak{PO}^p}{\partial x_i}|_{b=b_0}\boldsymbol{e}_j - \frac{\partial \mathfrak{PO}^p}{\partial x_j}|_{b=b_0}\boldsymbol{e}_i \\
\delta_{b_0}(\boldsymbol{e}_i \cup \boldsymbol{e}_j \cup \boldsymbol{e}_k) &= \frac{\partial \mathfrak{PO}^p}{\partial x_1}|_{b=b_0}\boldsymbol{e}_2 \cup \boldsymbol{e}_3 - \frac{\partial \mathfrak{PO}^p}{\partial x_2}|_{b=b_0}\boldsymbol{e}_1 \cup \boldsymbol{e}_3 + \\
&+ \frac{\partial \mathfrak{PO}^p}{\partial x_3}|_{b=b_0}\boldsymbol{e}_1 \cup \boldsymbol{e}_2 + \text{"extra terms"} \\
&\cdots \\
\delta_{b_0}(\boldsymbol{e}_1 \cup \cdots \cup \boldsymbol{e}_n) &= ??? \ .
\end{aligned} \tag{6.91}$$

for $i, j, k \in \{1, ..., n\}$ and $\{\boldsymbol{e}_1, ..., \boldsymbol{e}_n\}$ being a basis of $H^1(L(p) \cong T^n, \mathbb{Z}) \cong \mathbb{Z}^n$. Problems already arise when one wants to compute $\delta_{b_0}(\boldsymbol{e}_i \cup \boldsymbol{e}_j \cup \boldsymbol{e}_k)$ for Lagrangian tori of dimension $n \geq 3$, namely we yet do not know how to compute the appearing "extra terms".
We refer to Remark 6.4 where we again pick up this unsolved difficulty, show where exactly the problems arise and give a suggestion for further studies how to perhaps solve it once.

For proving the existence of a balanced Lagrangian torus the reader is referred to Proposition 4.7. of [FOOO2].
Our focus lies more on highlighting how an appropriate interior point $p_0 \in \mathring{\Delta}$ (and thus $L(p_0)$) can actually be detected and especially how to construct a (weak) Maurer-Cartan solution

$$b \in H^1(L(p_0); \Lambda_+) \tag{6.92}$$

thereof. In this context we further try to clarify why there is this extra specification

$$\rho : H_1(L(p_0); \mathbb{Z}) \to \mathbb{C}^* \tag{6.93}$$

in

$$HF((L(p_0), \rho, b), (L(p_0), \rho, b); \Lambda_0) \ . \tag{6.94}$$

To be more precise the stated Proposition 4.7. states that for a compact Fano toric manifold it exists $p_0 \in \mathring{\Delta}$ so that one finds

$$y_0 = (y_{0_1}, ..., y_{0_n}) \in \Lambda_0^n - \{0\} \tag{6.95}$$

such that

$$\frac{\partial \mathfrak{PO}^{p_0}}{\partial y_i}|_{y=y_0} \equiv 0 \tag{6.96}$$

for $i \in \{1, ..., n\}$ (remark that we can interpret $\mathfrak{PO}^{p_0}$ as a function depending on y when setting $y_j := e^{x_j} = \sum\limits_{k=0}^{\infty} \frac{x_j^k}{k!} \in \Lambda_0$ as in (5.171)).
For such an element y_0 we define

$$x_{0_i} = \underbrace{x_{0_i}^0}_{\in \mathbb{C}} + \underbrace{x_{0_i}^+}_{\in \Lambda_+} \in \Lambda_0 \tag{6.97}$$

via

$$y_{0_i} = e^{x_{0_i}^0 + x_{0_i}^+} = \underbrace{e^{x_{0_i}^0}}_{\mathbb{C}^*} \underbrace{\sum_{k=0}^{\infty} \frac{(x_{0_i}^+)^k}{k!}}_{\in \Lambda_0} . \tag{6.98}$$

Since x_{0_i} is unique up to addition of $2\pi i \mathbb{Z}$ we fix the convention that its imaginary part is contained in $[0, 2\pi)$ and get uniqueness.
By using the basis $\{e_1, ..., e_n\}$ (dual to the given basis $\{e_1^*, ..., e_n^*\}$ of N) of $H^1(L(p_0); \mathbb{Z})$ we get an element

$$x_0 = \sum_{i=1}^{n} x_{0_i} e_i \in H^1(L(p_0); \Lambda_0) . \tag{6.99}$$

Remark that $y_{0_i} \in \Lambda_0$ can be decomposed as

$$y_{0_i} = \underbrace{y_{0_i}^0}_{\in \mathbb{C}^*} + \underbrace{y_{0_i}^+}_{\in \Lambda_+} \in \Lambda_0 . \tag{6.100}$$

In the case $y_{0_i}^0 = 1$ for all i the construction above yields that $x_{0_i} \in \Lambda_+$ implying that (6.99) already defines a weak Maurer-Cartan solution

$$b = \sum_{i=1}^{n} x_{0_i} e_i \in H^1(L(p_0); \Lambda_+) \underbrace{\hookrightarrow}_{\text{Prop. 5.5}} \mathcal{M}_{\text{weak}}(L(p)) . \tag{6.101}$$

Unfortunately $y_{0_i}^0 = 1$ and therefore $x_{0_i}^0 = \ln(y_{0_i}^0) = 0$ does not hold in general. This is the point where the homomorphism

$$\begin{aligned} \rho : H_1(L(p_0); \mathbb{Z}) &\to \mathbb{C}^* \\ e_i^* &\mapsto y_{0_i} \end{aligned} \tag{6.102}$$

comes into play. It allows to "twist" the A_∞-algebra via the replacement

$$m_{k,\beta} \to m_{k,\beta}^{\rho} := \rho(\partial\beta) \cdot m_{k,\beta} . \tag{6.103}$$

In section 12 of [FOOO2] it is proven that

- $(H^*(L(p); \Lambda_0), m^\rho)$ is a filtered A_∞-algebra with unit PD$[L(p)]$.
- We have $H^1(L(p); \Lambda_+) \hookrightarrow \mathcal{M}_{\text{weak}}((H^*(L(p); \Lambda_0), m^\rho))$ and therefore the twisted potential function
$$\mathfrak{PO}_\rho^p : H^1(L(p); \Lambda_+) \to \Lambda_+ \tag{6.104}$$
is defined.

- For $x^0_{0_i} = \ln(y^0_{0_i})$ the identity

$$\mathfrak{PO}^{p_0}_{\rho}(\sum_{i=1}^{n}(\underbrace{x_{0_i} - x^0_{0_i}}_{\in H^1(L(p_0);\Lambda_+)})e_i) = \mathfrak{PO}^{p_0}(\sum_{i=1}^{n}\underbrace{x_{0_i}}_{\in H^1(L(p_0);\Lambda_0)} e_i) \tag{6.105}$$

provides that the potential function extends to a function defined on

$$H^1(L(p_0);\Lambda_0) \hookrightarrow \mathcal{M}_{\text{weak}}(L(p_0)) \tag{6.106}$$

and thus

$$\mathfrak{PO}^{p_0} : (\Lambda_0)^n \to \Lambda_+ \tag{6.107}$$

is defined.

In summary we want to highlight the essence of the considerations of the last pages. For elements of the form

$$b := \sum_{i=1}^{n} x_{0_i} e_i \in H^1(L(p_0);\Lambda_0) \tag{6.108}$$

for $y_{0_i} = e^{x_{0_i}}$ satisfying (6.96), we know that Lagrangian Floer Cohomology is defined (since b is a weak Maurer-Cartan solution).
Even more we get that $HF^*((L(p_0),\rho,b),(L(p_0),\rho,b);\Lambda_0)$ is non-trivial, meaning that we have an isomorphism of the form (6.86) that justifies $L(p_0)$ to be called balanced.
This non-triviality statement can be seen as follows:

Proof of Proposition 6.1 : For the following we always pretend that the starting equation

$$\frac{\partial \mathfrak{PO}^p}{\partial x_i}|_{b=b_0} = \frac{\partial \mathfrak{PO}^p}{\partial y_i}|_{b=b_0} \cdot y_i \tag{6.109}$$

($= 0$ for arbitrary dimensions (see (6.96)) or more general $\in \Lambda_+$ for the $n = 2$ case (see (6.90))) gives rise to a $b_0 \in H^1(L(p_0);\Lambda_+)$ meaning that we do not have to worry about the stated twisting process $m \to m^\rho$. The proof would work analogously without this requirement but appears less transparent since, due to the arising ρ terms, notations get more complicated. In the presented examples of section 6.2.2 we are always allowed to choose $b_0 = 0 \in H^1(L(p_0);\Lambda_+)$. Hence for our purposes this simplification is justified but the interested reader is referred to section 13 of [FOOO1] for details.

For such a b_0 we clearly have $\delta_{b_0}(\mathbf{e}_0) = 0$, since $H^k(T^n;\Lambda_0) = 0$ for $k < 0$, for $\mathbf{e}_0 \equiv PD([L(p)])$ being a generator of $H^0(L(p);\Lambda_0)$.

For general $n \in \mathbb{N}$ we continue by calculating

$$\begin{aligned} \delta_{b_0} : H^1(L(p);\Lambda_0) &\to H^0(L(p);\Lambda_0) \\ \mathbf{e}_i &\mapsto \cdots \mathbf{e}_0 \end{aligned} \tag{6.110}$$

for $i \in \{1, ..., n\}$ and $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ denoting a basis of $H^1(T^n; \Lambda_0)$.
Recall the defining equation for the potential function

$$\mathfrak{PO}(b) \cdot \underbrace{PD([L(p)])}_{= \, \mathbf{e}_0} = m(e^b) = \sum_{k=0}^{\infty} m_k(b, ..., b) \ . \tag{6.111}$$

For a fixed interior point $p_0 \in \mathring{\Delta}$ take the derivatives with respect to x_i evaluated at b_0 (recall $b = \sum_{i=1}^{n} x_i \mathbf{e}_i$)

$$\begin{aligned} \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_i}|_{b=b_0} \cdot \mathbf{e}_0 &= \frac{\partial}{\partial x_i}|_{b=b_0} \sum_{k=0}^{\infty} m_k(b, ..., b) = \\ &= \sum_{k_1,k_2=0}^{\infty} m_k(\underbrace{b_0, ..., b_0}_{k_1}, \mathbf{e}_i, \underbrace{b_0, ..., b_0}_{k_2}) \underbrace{=}_{\text{Def. 3.6}} \\ &= m_1^{b_0}(\mathbf{e}_i) \underbrace{=}_{\text{Prop. 3.2}} \delta_{b_0}(\mathbf{e}_i) \ . \end{aligned} \tag{6.112}$$

That is the first and second equation of (6.90) is proven. Remark that if we take δ_{b_0} with $b_0 = \sum_{i=1}^{n} x_{0_i} \mathbf{e}_i$ for x_{0_i} arising of $y_{0_i} = e^{x_{0_i}}$ for y_0 satisfying

$$\frac{\partial \mathfrak{PO}^{p_0}}{\partial y_i}|_{y=y_0} \equiv 0 \tag{6.113}$$

as in (6.96) we get that $\delta_{b_0}(\mathbf{e}_i) = 0$.
For such a pair p_0, b_0, giving rise to a disappearance of

$$\delta_{b_0} : H^1(L(p); \Lambda_0) \to H^0(L(p); \Lambda_0) \ , \tag{6.114}$$

let us think about how

$$\begin{aligned} \delta_{b_0} : H^i(L(p_0); \Lambda_0) &\to H^{i-1}(L(p_0); \Lambda_0) \\ \mathbf{x} &\mapsto \ \cdots \end{aligned} \tag{6.115}$$

can be computed.
Following the ideas of [FOOO1] we prove via induction (the base case is already done above) over (deg $\mathbf{x}, \mu(\beta)$) that all coboundary operations vanish then. Since a basis of $H^i(L(p_0); \Lambda_0)$ arises by cupping appropriate elements $\mathbf{e}_i$ of the basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ of $H^1(L(p_0); \Lambda_0)$ we can write

$$\mathbf{x} = \mathbf{x}_1 \cup \mathbf{x}_2 \underbrace{=}_{(5.189)} m_{2,\beta_0}(\mathbf{x}_1, \mathbf{x}_2) \tag{6.116}$$

with deg $\mathbf{x}_{1,2} <$ deg $\mathbf{x} = i$. Thus (6.115) writes as

$$\delta_{b_0}(\mathbf{x}) = \sum_{\beta} \underbrace{m_{1,\beta}^{b_0}(m_{2,\beta_0}(\mathbf{x}_1, \mathbf{x}_2))}_{\in \ H^{i+n-(n+\mu(\beta)+2-3)}(\ldots) = H^{(i+1)-\mu(\beta)}(\ldots)} T^{\frac{\omega(\beta)}{2\pi}} \ . \tag{6.117}$$

Remark that by degree reasons we only sum over classes β with (even) Maslov indices $\mu(\beta) \leq i+1$. Continuing with the inductive step we get

$$\begin{aligned} m_{1,\beta}^{b_0}(m_{2,\beta_0}(\mathbf{x}_1,\mathbf{x}_2)) \underbrace{=}_{\substack{A_\infty\text{-rel.}\\(5.112)}} & \sum_{\beta_1+\beta_2=\beta} (-1)^{\cdots} m_{2,\beta_2}(m_{1,\beta_1}^{b_0}(\mathbf{x}_1),\mathbf{x}_2)+ \\ & + \sum_{\beta_1+\beta_2=\beta} (-1)^{\cdots} m_{2,\beta_2}(\mathbf{x}_1, m_{1,\beta_1}^{b_0}(\mathbf{x}_2))+ \\ & + \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_2\neq 0}} (-1)^{\cdots} m_{1,\beta_1}^{b_0}(m_{2,\beta_2}(\mathbf{x}_1,\mathbf{x}_2)) = 0 \ . \end{aligned} \tag{6.118}$$

Here all three summands vanish since for the first and the second we have

$$\deg \mathbf{x}_{1,2} < \deg \mathbf{x} \tag{6.119}$$

and for the third

$$\mu(\beta_1) < \mu(\beta_1) + \underbrace{\mu(\beta_2)}_{<0} = \mu(\beta) \tag{6.120}$$

holds. For the fact $\beta_2 \neq 0$ implying $0 < \mu(\beta_2)$ recall Proposition 5.3 (i).
Summarizing the achieved results we conclude that for the pair

$$p_0 \in \mathring{\Delta}, b_0 \in H^1(L(p_0);\Lambda_+) \tag{6.121}$$

($L(p_0)$ being the balanced fiber, b_0 arising as a critical point of $\mathfrak{PO}^{p_0}$) the assertion about balanced fibers (and thus $HF^*(\cdots) \cong H^*(T^n;\Lambda_0)$) and (6.90) (in the special case that all derivatives vanish) is proven.

Unfortunately the presented consideration mostly provides just one fiber for which the Lagrangian Floer Cohomology can be computed.
The result can be improved for the $n=2$ case, that is examining 2 dimensional torus fibers in 4 dimensional toric manifolds. Precisely speaking it only remains to prove the third equation of (6.90). Analogously to above this can be done as follows

$$\delta_{b_0}(\underbrace{\mathbf{e}_1 \cup \mathbf{e}_2}_{=\,\mathbf{e}_{12}}) = \sum_\beta m_{1,\beta}^{b_0}(m_{2,\beta_0}(\mathbf{e}_1,\mathbf{e}_2)) T^{\frac{\omega(\beta)}{2\pi}} \quad \in H^{3-\mu(\beta)}(L(p);\Lambda_0) \ . \tag{6.122}$$

Again using the degree argument and (5.189) we conclude that only curves of class β with $\mu(\beta)=2$ have to be taken into account. Hence by using the A_∞-algebra relation (6.122) writes as

$$\begin{aligned} \delta_{b_0}(\mathbf{e}_{12}) = \sum_\beta \Big(& \sum_{\beta_1+\beta_2=\beta} (-1)^{\cdots} m_{2,\beta_2}(m_{1,\beta_1}^{b_0}(\mathbf{e}_1),\mathbf{e}_2) T^{\frac{\omega(\beta_1)+\omega(\beta_2)}{2\pi}} + \\ & + \sum_{\beta_1+\beta_2=\beta} (-1)^{\cdots} m_{2,\beta_2}(\mathbf{e}_1, m_{1,\beta_1}^{b_0}(\mathbf{e}_2)) T^{\frac{\omega(\beta_1)+\omega(\beta_2)}{2\pi}} + \\ & + \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_2\neq 0}} (-1)^{\cdots} m_{1,\beta_1}^{b_0}(m_{2,\beta_2}(\mathbf{e}_1,\mathbf{e}_2)) T^{\frac{\omega(\beta_1)+\omega(\beta_2)}{2\pi}} \Big) \ . \end{aligned} \tag{6.123}$$

For the first and the second summand we can use (6.112) and the fact that $\mathbf{e}_0 = PD([L(p)])$ is an A_∞-algebra unit (see Definition 3.5 (d)). We conclude

$$\begin{aligned} \delta_{b_0}(\mathbf{e}_{12}) = & \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_1}|_{b=b_0} \cdot m_2(\mathbf{e}_0, \mathbf{e}_2) + \\ & + \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_2}|_{b=b_0} \cdot m_2(\mathbf{e}_1, \mathbf{e}_0) + \\ & + \sum_{\substack{\beta_2 \\ \beta_2 \neq 0}} (-1)^{\cdots} d(m_{2,\beta_2}(\mathbf{e}_1, \mathbf{e}_2)) T^{\frac{\omega(\beta_2)}{2\pi}} = \\ \underbrace{=}_{\deg \mathbf{e}_1 = 1} & \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_1}|_{b=b_0} \cdot \mathbf{e}_2 - \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_2}|_{b=b_0} \cdot \mathbf{e}_1 \end{aligned} \tag{6.124}$$

which forms the end of the proof. ■

Remark 6.4. *Tying up with the unsolved difficulty raised in Remark 6.3 we highlight that the method above of calculating the Lagrangian Floer Cohomology for a continuum of Lagrangian torus fibers (not just a balanced one) can not easily be carried on for $n \geq 3$ cases. Already for $n = 3$ we have*

$$\delta_{b_0}(\underbrace{\boldsymbol{e}_1 \cup \boldsymbol{e}_2 \cup \boldsymbol{e}_3}_{=\ \boldsymbol{e}_{123}}) = \sum_\beta m_{1,\beta}^{b_0}(m_{2,\beta_0}(\boldsymbol{e}_1, \boldsymbol{e}_2 \cup \boldsymbol{e}_3)) T^{\frac{\omega(\beta)}{2\pi}} \quad \in H^{4-\mu(\beta)}(L(p); \Lambda_0) \tag{6.125}$$

that is additionally classes β with $\mu(\beta) = 4$ have to be taken into account. This yields

$$\delta_{b_0}(\boldsymbol{e}_{123}) = \underbrace{\sum_{\substack{\beta \\ \mu(\beta)=2}} m_{1,\beta}^{b_0}(m_{2,\beta_0}(\boldsymbol{e}_1, \boldsymbol{e}_2 \cup \boldsymbol{e}_3)) T^{\frac{\omega(\beta)}{2\pi}}}_{(I)} + \underbrace{\sum_{\substack{\beta \\ \mu(\beta)=4}} m_{1,\beta}^{b_0}(m_{2,\beta_0}(\boldsymbol{e}_1, \boldsymbol{e}_2 \cup \boldsymbol{e}_3)) T^{\frac{\omega(\beta)}{2\pi}}}_{(II)} . \tag{6.126}$$

For (I) we proceed as above by using the A_∞-relation and (5.189) which yields

$$\begin{aligned} (I) = & \sum_\beta \Big(\sum_{\beta_1+\beta_2=\beta} (-1)^{\cdots} m_{2,\beta_2}(m_{1,\beta_1}^{b_0}(\boldsymbol{e}_1), \boldsymbol{e}_2 \cup \boldsymbol{e}_3) T^{\frac{\omega(\beta_1)+\omega(\beta_2)}{2\pi}} + \\ & + \sum_{\beta_1+\beta_2=\beta} (-1)^{\cdots} m_{2,\beta_2}(\boldsymbol{e}_1, m_{1,\beta_1}^{b_0}(\boldsymbol{e}_2) \cup \boldsymbol{e}_3) T^{\frac{\omega(\beta_1)+\omega(\beta_2)}{2\pi}} + \\ & + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2 \neq 0}} (-1)^{\cdots} m_{1,\beta_1}^{b_0}(m_{2,\beta_2}(\boldsymbol{e}_1, \boldsymbol{e}_2 \cup \boldsymbol{e}_3)) T^{\frac{\omega(\beta_1)+\omega(\beta_2)}{2\pi}} \Big) = \\ = & \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_1}|_{b=b_0} \cdot \boldsymbol{e}_2 \cup \boldsymbol{e}_3 - \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_2}|_{b=b_0} \cdot \boldsymbol{e}_1 \cup \boldsymbol{e}_3 + \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_3}|_{b=b_0} \cdot \boldsymbol{e}_1 \cup \boldsymbol{e}_2 . \end{aligned} \tag{6.127}$$

Analogously we get for classes β of Maslov index $\mu(\beta) = 4$ (remark that for clarity's *sake we always neglect the appearing* $T^{\frac{\omega(\cdot)}{2\pi}}$ *terms)*

$$
\begin{aligned}
(II) = \cdots = & \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_1}|_{b=b_0} \cdot \sum_{\substack{\beta_2 \\ \mu(\beta_2)=2}} m_{2,\beta_2}(\boldsymbol{e}_0, \boldsymbol{e}_2 \cup \boldsymbol{e}_3) + \sum_{\substack{\beta_1 \\ \mu(\beta_1)=4}} \underbrace{m^{b_0}_{1,\beta_1}(\boldsymbol{e}_1)}_{=\,0} \cup \boldsymbol{e}_2 \cup \boldsymbol{e}_3 + \\
& + \sum_{\substack{\beta_2 \\ \mu(\beta_2)=2}} \left(\frac{\partial \mathfrak{PO}^{p_0}}{\partial x_2}|_{b=b_0} \cdot m_{2,\beta_2}(\boldsymbol{e}_1, \boldsymbol{e}_3) - \frac{\partial \mathfrak{PO}^{p_0}}{\partial x_3}|_{b=b_0} \cdot m_{2,\beta_2}(\boldsymbol{e}_1, \boldsymbol{e}_2) \right) + \\
& \qquad + \sum_{\substack{\beta_1 \\ \mu(\beta_1)=4}} \boldsymbol{e}_1 \cup \underbrace{m^{b_0}_{1,\beta_1}(\boldsymbol{e}_2 \cup \boldsymbol{e}_3)}_{=\,0} + \\
& + \sum_{\beta} \sum_{\substack{\beta_1+\beta_2=\beta \\ \mu(\beta_i)=2}} m^{b_0}_{1,\beta_1}(m_{2,\beta_2}(\boldsymbol{e}_1, \boldsymbol{e}_2 \cup \boldsymbol{e}_3)) \ .
\end{aligned}
\tag{6.128}
$$

So for computing these so called "extra terms" we have to make sense of terms of the form

$$m_{2,\beta}(\boldsymbol{e}_i, \boldsymbol{e}_j) \quad \textit{and} \quad m^{b_0}_{1,\beta_1}(m_{2,\beta_2}(\boldsymbol{e}_i, \boldsymbol{e}_j \cup \boldsymbol{e}_k)) \ . \tag{6.129}$$

We believe that this can be done quite similar to how we computed

$$m_{l,\beta}(x, ..., x) \quad \textit{for} \quad x \in H^1(L(p); \Lambda_0^{\mathbb{R}}) \tag{6.130}$$

in the proof of Proposition 5.5 but unfortunately could not prove it yet.
When examining tori of dimension $n \in \{4, 5, 6, ...\}$ *things even get less transparent since now, by degree reasons, curves of Maslov index* $\mu(\beta) \in \{5, 6, 7\} \cap 2\mathbb{Z}$ *have to be considered.*

6.2.2 Examples: (Non-)displacement results for Lagrangian tori in $S^2_{r_1} \times S^2_{r_2}$ and $\mathbb{C}P^2$

The last section's aim was to present a method of how one can compute

$$HF((L(p), b), (L(p), b); \Lambda_0) \tag{6.131}$$

at least for some Lagrangian tori $L(p)$ in M. Remark that it does not matter if we see $L(p)$ as one Lagrangian, and describe the setup via an A_∞-algebra, or if we regard it as a pair $L_1 = L_2 = L(p)$, and use the concepts of A_∞-bimodules. As highlighted in (6.79) the thereof arising Lagrangian Floer Cohomologies are identical.
Here we explicitly discuss the presented concepts for M^4 being either $S^2_{r_1} \times S^2_{r_2}$ or $\mathbb{C}P^2$, which are both Fano toric. Thanks to (6.90) this setup even provides a continuum of 2 dimensional tori for which $HF(\cdots)$ is computable.
Then we are finally in the position to answer question (1.3), posed in the introductory chapter 1, namely to derive a lower bound for the number of intersection points of $L(p)$ and $\psi(L(p))$ for ψ being a Hamiltonian diffeomorphism.

As usually for smooth maps $H : [0,1] \times M \to \mathbb{R}$ we have $\phi_H^t(\cdot)$ as the solution of $\dot{x}(t) = X_H(t, x(t))$ for X_H being the Hamiltonian vector field of H.
In order to find such a lower bound estimate we rely on Theorem J of [FOOO1], that we want to state here without proving it:

Theorem 6.2

Let L be a relatively spin Lagrangian submanifold in M and ψ be a Hamiltonian diffeomorphism with Hofer norm

$$\mu = \|\psi\| = \inf_{\substack{H;\\ \phi_H^1=\psi}} \int_0^1 (max\ H(t,\cdot) - min\ H(t,\cdot))dt \tag{6.132}$$

such that $\psi(L)$ is transversal to L. For $b \in \mathcal{M}_{weak}(L)$ and a Lagrangian Floer cohomology of the form

$$HF((L(p),b),(L(p),b);\Lambda_0) \cong (\Lambda_0)^m \oplus \bigoplus_{i=1}^n \frac{\Lambda_0}{T^{\lambda_i}\Lambda_0} \tag{6.133}$$

we have

$$\#\{p \in \psi(L) \cap L\} \geq m + 2n(\mu) \tag{6.134}$$

for

$$n(\mu) = \#\{i|\lambda_i \geq \mu\}\ . \tag{6.135}$$

(a) (Non-)displaceability of Lagrangian $2-$tori in $S^2_{r_1} \times S^2_{r_2}$:

Without loss of generality we assume $r_1 \leq r_2$ in the following. Recall section 5.1 where we derived the moment polytope for this toric symplectic manifold. By (5.19) we know that it is of the form

$$\begin{aligned}\Delta^{2,4}_{\lambda_{1,2}=0,\lambda_3=-2r_1,\lambda_4=-2r_2} &= \{p \in \mathbb{R}^2 \mid \langle p, e_i\rangle \geq 0; \langle p, -e_1\rangle \geq -2r_1; \langle p, -e_2\rangle \geq -2r_2\} \\ &= \{p \in \mathbb{R}^2 \mid l_{1,2}(p) = p_{1,2} \geq 0;\ l_3(p) = 2r_1 - p_1 \geq 0; \\ &\qquad l_4(p) = 2r_2 - p_2 \geq 0\}\end{aligned} \tag{6.136}$$

with inward pointing normal vectors

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = -e_1, \quad v_4 = -e_2\ . \tag{6.137}$$

Due to Proposition 5.5 we know that the potential function thus writes as

$$\begin{aligned}\mathfrak{PO}^p : (\Lambda_+)^2 &\to \Lambda_+ \\ x &\mapsto e^{x_1}T^{p_1} + e^{x_2}T^{p_2} + e^{-x_1}T^{2r_1-p_1} + e^{-x_2}T^{2r_2-p_2}\ .\end{aligned} \tag{6.138}$$

Taking the derivatives with respect to x_1, x_2 at the point $b_0 = (0,0) \in H^1(L(p);\Lambda_+)$ yields

$$\begin{aligned}\frac{\partial \mathfrak{PO}^p}{\partial x_1}|_{b=b_0} &= T^{p_1} - T^{2r_1-p_1} = 0 \\ \frac{\partial \mathfrak{PO}^p}{\partial x_2}|_{b=b_0} &= T^{p_2} - T^{2r_2-p_2} = (\underbrace{1-T^{2r_2-2p_2}}_{\in\ \Lambda_0})T^{p_2}\end{aligned} \tag{6.139}$$

for a continuum of Lagrangian torus fibers $L(p) \cong T^2$ over $\{p \in \mathring{\Delta} |\, p_1 = r_1,\ p_2 < r_2\}$. To compute the Lagrangian Floer Cohomology we have to consider the sequence

$$0 \longrightarrow H^2(L(p);\Lambda_0) \xrightarrow{\delta_{b_0}} H^1(L(p);\Lambda_0) \xrightarrow{\delta_{b_0}} H^0(L(p);\Lambda_0) \longrightarrow 0 \tag{6.140}$$

that can be written as, when applying the results of Proposition 6.1,

$$\begin{array}{ccccc} 0 \longrightarrow \Lambda_0 \longrightarrow & (\Lambda_0)^2 & \longrightarrow & \Lambda_0 & \longrightarrow 0 \\ \mathbf{e}_1 \cup \mathbf{e}_2 \longmapsto & (-1+T^{2r_2-2p_2})T^{p_2}\mathbf{e}_1 & & & \\ & & \mathbf{e}_1 \longmapsto & 0 & \\ & & \mathbf{e}_2 \longmapsto & (1-T^{2r_2-2p_2})T^{p_2}\mathbf{e}_0\ . & \end{array} \tag{6.141}$$

We conclude that the Lagrangian Floer cohomology is of the form

$$HF^*((L(p),b_0),(L(p),b_0);\Lambda_0) \cong \left(\frac{\Lambda_0}{T^{p_2}\Lambda_0}\right)^{\oplus 2}\ . \tag{6.142}$$

Remark that, for applying the stated Theorem 6.2, in the definition of the filtered A_∞-algebra (Theorem 5.3) we substituted

$$T \to T^{\frac{1}{2\pi}} \tag{6.143}$$

compared to K. Fukaya's Definition in [FOOO1]. This simplifies appearing terms in a way such that one gets rid of the 2π factors when inserting $\omega(\beta_i) = 2\pi l_i(p)$. When now substituting back (6.142) reads as

$$HF^*((L(p),b_0),(L(p),b_0);\Lambda_0) \cong \left(\frac{\Lambda_0}{T^{2\pi p_2}\Lambda_0}\right)^{\oplus 2}\ . \tag{6.144}$$

With Theorem 6.2 we finally conclude

$$\#\{q \in \psi(L(p)) \cap L(p)\} \geq 4 = 2^{n=2} \tag{6.145}$$

for Hamiltonian diffeomorphism

$$\psi : S^2_{r_1} \times S^2_{r_2} \to S^2_{r_1} \times S^2_{r_2} \tag{6.146}$$

with Hofer norm

$$\|\psi\| < 2\pi p_2 \tag{6.147}$$

such that $\psi(L(p))$ intersects $L(p)$ transversally.
Remark that this argumentation works analogously, and thus the number of intersection points is bigger or equal 4, for $L(p)$ over $\{p \in \mathring{\Delta} | \ p_1 < r_1, \ p_2 = r_2\}$ and $\|\psi\| < 2\pi p_1$.
To tie up with the considerations in Proposition 6.1 about critical points of $\mathfrak{PO}$ we remark that for $b_0 = 0$ the only balanced fiber is given by

$$L_0 := \mu^{-1}((r_1, r_2)) \tag{6.148}$$

resulting in

$$\frac{\partial \mathfrak{PO}^p}{\partial x_i}|_{b=b_0} = 0 \quad \text{for} \quad i \in \{1, 2\} \tag{6.149}$$

implying

$$HF^*((L_0, b_0), (L_0, b_0); \Lambda_0) \cong (\Lambda_0)^{\oplus 2} \ . \tag{6.150}$$

With Theorem 6.2 this means that L_0 is indeed non-displaceable (or equivalently the displacement energy is ∞), that is we always find intersection points no matter how big the Hofer energy of the Hamiltonian diffeomorphisms $\|\psi\|$ is.

(b) Visualizing $T^1 = S^1$ in $S^2_{r_1}$ (example (a) for $n = 1$) :

Clearly this particular case is just a simplification of the results achieved in example (a) and yields nothing new. Nevertheless due to its transparency it well illustrates the usefulness of Lagrangian Floer Cohomology combined with Theorem 6.2 to get an lower bound on the number of intersections points.
The corresponding moment polytope

$$\begin{aligned} \Delta^{1,2}_{\lambda_1=0,\lambda_2=-2r_1} &= \{p \in \mathbb{R}^1 \mid \langle p, e_1\rangle \geq 0; \langle p, -e_1\rangle \geq -2r_1\} \\ &= \{p \in \mathbb{R}^2 \mid l_1(p) = p_1 \geq 0; \ l_2(p) = 2r_1 - p_1 \geq 0\} \end{aligned} \tag{6.151}$$

yields a potential function of the form

$$\frac{\partial \mathfrak{PO}^p}{\partial x_1}|_{b=b_0} = T^{p_1} - T^{2r_1 - p_1} = (\underbrace{1 - T^{2r_1 - 2p_1}}_{\in \ \Lambda_0})T^{p_1} \tag{6.152}$$

for $b_0 = 0 \in H^1(L(p); \Lambda_+)$ and fibers $L(p)$ over $\{p \in \mathring{\Delta} | \ p_1 \leq r_1\}$.
Performing similar considerations as in the previous example we get

$$HF^*((L(p), b_0), (L(p), b_0); \Lambda_0) \cong \begin{cases} \frac{\Lambda_0}{T^{2\pi p_1}\Lambda_0} & , \text{ for } p_1 < r_1 \\ \Lambda_0 & , \text{ for } p_1 = r_1 \end{cases} . \tag{6.153}$$

Again with Theorem 6.2 we get for $L(p_1 < r_1)$

$$\#\{q \in \psi(L(p)) \cap L(p)\} \geq 2 = 2^{n=1} \tag{6.154}$$

for Hamiltonian diffeomorphism

$$\psi : S^2_{r_1} \to S^2_{r_1} \tag{6.155}$$

with Hofer norm

$$\|\psi\| < 2\pi p_1 \tag{6.156}$$

such that $\psi(L(p))$ intersects $L(p)$ transversally. Analogously $L(r_1)$ is the only non-displaceable fibre for $b_0 = 0$.
Pictorially we can visualize the setup as in figure (6.2) where the balanced Lagrangian tori $L_0 = S^1_{\text{equ.}} = L(r_1)$ is drawn in blue. The picture well illustrates the

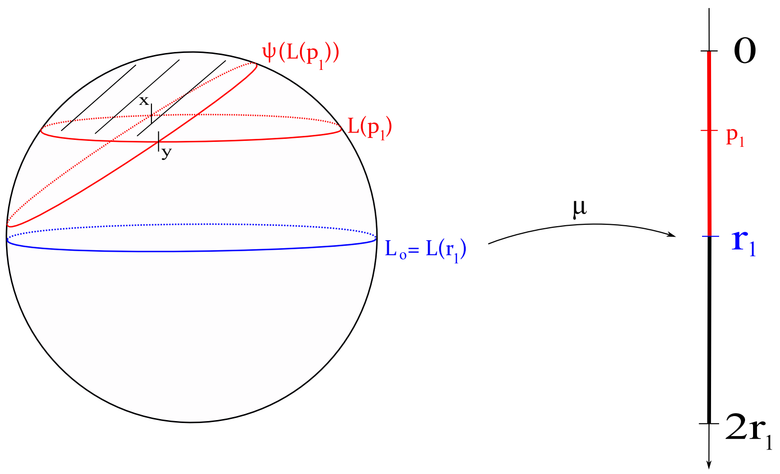


Figure 6.2: Intersections of Lagrangian tori $L(p_1)$ in $S^2_{r_1}$

results previously achieved by using abstract Lagrangian Floer Cohomology. We easily see that if we move the Lagrangian $L(p_1)$, by using a Hamiltonian diffeomorphism with Hofer norm $\|\psi\| > 2\pi p_1$ (that is bigger than the volume $\text{vol}_{L(p_1)} = 2\pi p_1$ of the upper cap determined by its boundary $L(p_1)$), the dashed area between

$$\psi(L(p_1)) \quad \text{and} \quad L(p_1) \tag{6.157}$$

vanishes. This implies

$$\psi(L(p_1)) \cap L(p_1) = \emptyset\ . \tag{6.158}$$

Otherwise ($\|\psi\| < 2\pi p_1$) we always have $2^{n=1}$ intersection points x, y.
For L_0 (being the equator) we do not have such a restriction onto the Hofer norm, since for ψ with $\|\psi\| = 2\pi r_1$ we have $\psi(L_0) = L_0$, implying that L_0 is a non-displaceable Lagrangian tori in $S^2_{r_1}$.

(c) (Non-)displaceability of Lagrangian $2-$tori in $\mathbb{C}P^2$:

Similar to the way how we treated the toric manifold $T^2 \hookrightarrow S^2_{r_1} \times S^2_{r_2}$, we can handle the case for $\mathbb{C}P^2$.
Recall (5.31), namely its moment polytope is given by

$$\begin{aligned}\Delta^{2,3}_{\lambda_1=0,\lambda_2=0,\lambda_3=-1} &= \{p \in \mathbb{R}^2 \mid \langle p, e_1\rangle, \langle p, e_2\rangle \geq 0; \langle p, -e_1-e_2\rangle \geq -1\} = \\ &= \{p \in \mathbb{R}^2 \mid l_{1,2}(p) = p_{1,2} \geq 0;\ l_3(p) = 1 - p_1 - p_2 \geq 0\}\end{aligned} \tag{6.159}$$

with inward pointing normal vectors of the form

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = -e_1 - e_2 \ . \tag{6.160}$$

Proposition 5.5 allows to write the potential function in the form

$$\mathfrak{PO}^p(x) = e^{x_1} T^{p_1} + e^{x_2} T^{p_2} + e^{-x_1 - x_2} T^{1-p_1-p_2} \ . \tag{6.161}$$

Then its derivatives with respect to x_1, x_2 (at $b_0 = (0,0) \in H^1(L(p); \Lambda_+)$) are given by

$$\begin{aligned} \frac{\partial \mathfrak{PO}^p}{\partial x_1}\Big|_{b=b_0} &= T^{p_1} - T^{1-p_1-p_2} \\ \frac{\partial \mathfrak{PO}^p}{\partial x_2}\Big|_{b=b_0} &= T^{p_2} - T^{1-p_1-p_2} \ . \end{aligned} \tag{6.162}$$

When now considering Lagrangian torus fibers over

$$\{p \in \mathring{\Delta} |\ p_1 = 1/3,\ p_2 > 1/3\} \quad \text{or} \quad \{p \in \mathring{\Delta} |\ p_1 > 1/3,\ p_2 = 1/3\} \tag{6.163}$$

we either get for the first case

$$\begin{aligned} \frac{\partial \mathfrak{PO}^p}{\partial x_1}\Big|_{b=b_0} &= \overbrace{\left(T^{p_2 - 1/3} - 1\right)}^{\in\ \Lambda_0} T^{2/3 - p_2} \\ \frac{\partial \mathfrak{PO}^p}{\partial x_2}\Big|_{b=b_0} &= \underbrace{\left(T^{2p_2 - 2/3} - 1\right)}_{\in\ \Lambda_0} T^{2/3 - p_2} \end{aligned} \tag{6.164}$$

or

$$\begin{aligned} \frac{\partial \mathfrak{PO}^p}{\partial x_1}\Big|_{b=b_0} &= \overbrace{\left(T^{2p_1 - 2/3} - 1\right)}^{\in\ \Lambda_0} T^{2/3 - p_1} \\ \frac{\partial \mathfrak{PO}^p}{\partial x_2}\Big|_{b=b_0} &= \underbrace{\left(T^{p_1 - 1/3} - 1\right)}_{\in\ \Lambda_0} T^{2/3 - p_1} \end{aligned} \tag{6.165}$$

for the second.
For computing the Lagrangian Floer Cohomology we consider the sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Lambda_0 & \longrightarrow & & (\Lambda_0)^2 & & \longrightarrow & \Lambda_0 & \longrightarrow \quad 0 \\ & \mathbf{e}_1 \cup \mathbf{e}_2 & \longmapsto & & T^{2/3-p_i}((\cdots)\mathbf{e}_2 - (\cdots)\mathbf{e}_1) & & & & \\ & & & & & \mathbf{e}_1 & \longmapsto & (\cdots)\ T^{2/3-p_i}\mathbf{e}_0 & \\ & & & & & \mathbf{e}_2 & \longmapsto & (\cdots)\ T^{2/3-p_i}\mathbf{e}_0 & \end{array} \tag{6.166}$$

that yields (recall the resubstitution $T \to T^{2\pi}$)

$$HF^*((L(p), b_0), (L(p), b_0); \Lambda_0) \cong \left(\frac{\Lambda_0}{T^{2\pi(2/3-p_i)}\Lambda_0} \right)^{\oplus 2} \tag{6.167}$$

for $i = 2$ in the first respectively $i = 1$ in the second case. Hence with Theorem 6.2 we get:

$$\#\{q \in \psi(L(p)) \cap L(p)\} \geq 4 = 2^{n=2} \tag{6.168}$$

for Hamiltonian diffeomorphism

$$\psi : \mathbb{C}P^2 \to \mathbb{C}P^2 \tag{6.169}$$

with Hofer norm

$$\|\psi\| < 2\pi(2/3 - p_i) \tag{6.170}$$

such that $\psi(L(p))$ intersects $L(p)$ transversally. Here we mean

$$p_i = \begin{cases} p_1, & \text{for } p \in \{\mathring{\Delta} |\ p_1 > 1/3,\ p_2 = 1/3\} \\ p_2, & \text{for } p \in \{\mathring{\Delta} |\ p_1 = 1/3,\ p_2 > 1/3\} \end{cases} . \tag{6.171}$$

For b_0 remark that $L_0 := \mu^{-1}((1/3, 1/3))$ is the only non-displaceable fiber. This holds since in that case we have

$$\frac{\partial \mathfrak{PO}^p}{\partial x_i}|_{b=b_0} = 0 \tag{6.172}$$

which yields

$$HF^*((L_0, b_0), (L_0, b_0); \Lambda_0) \cong (\Lambda_0)^{\oplus 2} \tag{6.173}$$

and thus, with Theorem 6.2, we conclude

$$\psi(L) \cap L \neq \emptyset \tag{6.174}$$

for all Hamiltonian diffeomorphisms $\psi : \mathbb{C}P^2 \to \mathbb{C}P^2$.

6.2.3 Conclusion and suggestions for further studies

In summary it remains to say that presented theories and thereof arising methods provide an insight into the behavior of at least some (in the 2 dimensional case even for a continuum of) Lagrangian subtori $L \subset M$ when applying Hamiltonian diffeomorphisms ψ on Fano toric symplectic manifolds M.
Though we are not able to examine all Lagrangian subtori in $S^2_{r_1} \times S^2_{r_2}$, the presented method of K. Fukaya et al. improves a result of Y.V. Chekanov achieved in [Che]. The authors´ work extends the allowed bound on the Hofer norm for ψ from

$$\|\psi\| < 2\pi r_1 \quad \text{to} \quad \|\psi\| < 2\pi p_2 \tag{6.175}$$

with $r_1 < p_2 \leq r_2$ (recall that we required wlog. $r_1 \leq r_2$).
One goal for the future, as already announced in Remark 6.4, could be to further extend the achieved result (of Proposition 6.1) to a continuum of Lagrangian submanifolds of dimension $n = 3, 4, \ldots$.

A different (but mainly also pioneered by K. Fukaya in [Fu]) technique for the examination of Lagrangian submanifolds, by using holomorphic curve theory, is the incorporation of string topology. A first step towards that kind of treatment (of symplectic field theory) is discussed by K. Cieliebak and J. Latschev in [CL] for the case of cotangent bundles. In contrast to how we derived the A_∞ description by evaluating marked points of Σ, this new approach uses the evaluation of the whole boundary $\partial\Sigma$ and one thus deals with the free loop space of L. Here the appearing bubbling phenomena can be described and maybe well handled by string topology operations on L. It seems extremely interesting to explore the relation between the two possible ways of approaching, namely A_∞-structures (the content of this thesis) and string topology (the content of future research).

Chapter 7

Applications II: Properties of Ψ - Relation to physics

We are tying in with the considerations of section 6.2 about possible applications of the potential function $\mathfrak{PO}$ for mathematics, namely to use its derivatives to compute the coboundary operator, defining the Lagrangian Floer Cohomology. Here in contrast we directly pick up the considerations of the introductory chapter outlining the importance of A_∞-structures in string field theory. Precisely speaking we aim to provide a first feeling for the relevance of A_∞-algebra structures in order to describe tree-level scattering amplitudes of open string states. As we show in the upcoming sections the space of physical states can naturally be equipped with an A_∞-algebra structure. This algebraic description again leads to the definition of a potential function (the *superpotential* Ψ) out of the given A_∞-algebra structure. As in the case for toric symplectic manifolds we are interested in its derivatives and in particular critical points. As we see these are the states defining the string moduli space (the vacua of the underlying cubic string field theory) respectively are strict bounding cochains (recall Definition 3.7) when regarding things from a mathematical perspective.
We refer to [Laz], [Tom] that provided us a helpful approach to the way how physicists regard and work with A_∞-structures. These sources can in addition be seen as the main references we refer to in this chapter. For the appearing mathematics we again make use of K. Fukaya et al.s´ ideas presented in [FOOO1].

7.1 Basics of D-brane geometry (A side)

In this first section concepts are introduced in general D-brane fashion first. Since it is more related to the way how things are described in the preceding chapters, we outline their concrete A side realizations in brackets.

For the way how concepts appear in B side considerations, we refer to the literature especially [Laz].

As remarked in the introductory chapter A_∞-algebras generalize the concept of differential graded algebras (D.G.A.). Assume a given D.G.A.

$$(\mathcal{A} = \bigoplus_{m\in\mathbb{Z}} \mathcal{A}^m, \cdot, Q) \tag{7.1}$$

over a field R. Speaking in physical terms the $\mathbb{Z}$-graded vector space $\mathcal{A}$ defines the *Hilbert space of (off-shell) string states* of a string theory with the grading $|\cdot|$ denoting the *worldsheet degree*.
On the A side the physical target space consists of a Calabi-Yau threefold (M, ω, J, Ω). The notion of an A-type brane means a pair (L, E), where $L \subset M$ is a *special* ($\Omega|_L =$ const.) Lagrangian submanifold and E denoting a complex vector bundle over L. In this setup the $\mathbb{Z}$-graded complex vector space $\mathcal{A}$ is then given by

$$\mathcal{A} := \Omega^*(L, \mathrm{End}(E)) \ . \tag{7.2}$$

This viewpoint gets applied when considering only one single given A-brane. It generalizes from a D.G.A. to a categorical description, and thus from A_∞-algebras to A_∞-categories, when considering many A-branes at once. Here the branes L_i form the objects of this so called *Fukaya category* and morphisms $\mathrm{Hom}(L_i, L_j)$ are given by the Hilbert space of open strings stretching between the branes (realized by Lagrangian Floer Cohomology $HF(L_i, L_j)$ on the A side). Here the categorical description comes into play since multiplication respectively A_∞ homomorphisms are only defined if source and target brane coincide that is

$$m_n : \mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_2, L_3) \otimes ... \otimes \mathrm{Hom}(L_n, L_{n+1}) \to \mathrm{Hom}(L_1, L_{n+1}) \ . \tag{7.3}$$

We do not want to further develop this more general kind of description and refer the reader to [LazII] and [Tom] for details.

For the D.G.A. (7.1) the differential Q (the Dolbeault coboundary operator

$$\overline{\partial}_{\mathrm{End}(E)} : \Omega^{p,q}(L, \mathrm{End}(E)) \to \Omega^{p,q+1}(L, \mathrm{End}(E)) \tag{7.4}$$

on the A side) symbolizes the *BRST coboundary operator* and with $\cdot$ (realized by the ordinary wedge product $\wedge$) the *string product* is denoted.
A *cubic open string field theory* is characterized by an action functional of the form

$$\begin{aligned} S : \mathcal{A}^{m=1} &\to R \\ \phi &\mapsto \frac{1}{2}\langle \phi, Q\phi \rangle + \frac{1}{3}\langle \phi, \phi \cdot \phi \rangle \end{aligned} \tag{7.5}$$

defined on the space of *string fields* that is formed by homogeneous degree 1 elements $\mathcal{A}^1 \subset \mathcal{A}$. Here the bracket $\langle \cdot, \cdot \rangle$ denotes a non-degenerate bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \to R \tag{7.6}$$

satisfying

(i) $\langle u \cdot v, w \rangle = \langle u, v \cdot w \rangle$

(ii) $\langle Qu, v \rangle = (-1)^{|u|+1} \langle u, Qv \rangle$

(iii) $\langle u, v \rangle = (-1)^{|u||v|} \langle v, u \rangle$

(iv) $\langle u, v \rangle = 0 \quad \text{if} \quad |u| + |v| \neq 3$

for $u, v, w \in \mathcal{A}$. Item (iv) reflects the fact that we are dealing with a target space M being a CY threefold. The Lagrangian submanifolds are thus of real dimension 3. For A-type branes the bracket $\langle \cdot, \cdot \rangle$ is defined by wedging the forms u, v and then integrating its trace along L, that is

$$\langle u, v \rangle := \int_L tr_E(u \wedge v) \ . \tag{7.7}$$

To guarantee definedness of this pairing one needs to require $|u| + |v| = 3$. Remark that Q is of degree $+1$, string fields $\psi \in \mathcal{A}^1$ and thus both summands in (7.5) fulfill this requirement. Further (iv) simplifies (iii) from a graded symmetry to a symmetry condition

$$\langle u, v \rangle = \langle v, u \rangle \ . \tag{7.8}$$

Trivially properties (i)-(iii) are also fulfilled when setting $\langle \cdot, \cdot \rangle$ as in (7.7).
In that sense we further use an antilinear operator

$$c : \mathcal{A}^k \to \mathcal{A}^{3-k} \tag{7.9}$$

on $\mathcal{A}$ satisfying

$$\overline{\langle cu, v \rangle} = \langle cv, u \rangle \tag{7.10}$$

and $c^2 = \mathrm{Id}$. This in turn allows to define a Hermitian (due to (7.10)) product

$$\begin{aligned} h : \mathcal{A} \times \mathcal{A} &\to R \\ (u, v) &\mapsto \langle cu, v \rangle \ . \end{aligned} \tag{7.11}$$

Due to $|cu| + |v| = 3$ (iv) combined with $|cu| + |u| = 3$ (7.9) we have

$$h(u, v) = 0 \quad \text{if} \quad |u| \neq |v| \ . \tag{7.12}$$

Further

$$h(cu, cv) = \langle c^2 u, cv \rangle = \langle cv, u \rangle = h(v, u) = \overline{h(u, v)} \tag{7.13}$$

yields that the operator c is an antilinear isometry.
As maybe already expected for the A side the operator c is just the ordinary Hodge star operator $\overline{*}_{\mathrm{End}(E)}$.

The degree -1 operator Q^+ is defined by

$$Q^+ := (-1)^{|\cdot|} cQc \ (\overbrace{\equiv}^{A \text{ side}} \overline{\partial}^*_{\mathrm{End}(E)} = - \ \overline{*}_{\mathrm{End}(E)} \circ \overline{\partial}_{\mathrm{End}(E)} \circ \overline{*}_{\mathrm{End}(E)}) \ . \tag{7.14}$$

Here Q^+ denotes the Hermitian conjugate of Q with respect to h since

$$\begin{aligned} h(Q^+u,v) &= (-1)^{|u|}h(cQcu,v) = (-1)^{|u|}\langle Qcu,v\rangle = (-1)^{|u|+|cu|+1}\langle cu,Qv\rangle = \\ &= (-1)^4\langle cu,Qv\rangle = h(u,Qv) \ . \end{aligned} \tag{7.15}$$

Out of the definition of Q^+ we easily deduce

(i) $(Q^+)^2 = (-1)^{|\cdot|}Q^+(cQc) = (-1)^{2|\cdot|}cQccQc = cQ^2c = 0$

(ii) $(Q^+)^+ = (-1)^{|\cdot|}cQ^+c = (-1)^{2|\cdot|}ccQcc = Q$.

After introducing the needed mathematical operations we aim to describe the relevance of these notions when doing physics.
Remark that the physical laws of the theory are encoded in (7.5). It is invariant under certain gauge transformations and for the following we fix this symmetry by requiring

$$Q^+\phi = 0 \ . \tag{7.16}$$

As in the literature the thereof arising vector space

$$\ker Q \cap \ker Q^+ =: K \tag{7.17}$$

is used as the Hilbert space of *physical states*. Mathematically speaking of the A side this space K is formed by harmonic forms

$$\begin{aligned} \Omega^*_{\text{harm.}}(L,\text{End}(E)) := \{v \in \Omega^*(L,\text{End}(E)) | \Delta_{\text{End}(E)}v = \\ =(\overline{\partial}^*_{\text{End}(E)}\overline{\partial}_{\text{End}(E)} + \overline{\partial}_{\text{End}(E)}\overline{\partial}^*_{\text{End}(E)})v = 0\} \ . \end{aligned} \tag{7.18}$$

Using Hodge decomposition we continue by orthogonally decomposing (with respect to h) the space of off-shell states

$$\mathcal{A} = K \oplus \underbrace{\text{Im } Q \oplus \text{Im } Q^+}_{K^\perp} \tag{7.19}$$

with

$$\ker Q = K \oplus \text{Im } Q \quad \text{and} \quad \ker Q^+ = K \oplus \text{Im } Q^+ \tag{7.20}$$

where the elements of Im Q and Im Q^+ are called *spurious-* and *unphysical states* respectively. Remark that we therefore get

$$K \cong H^*_Q(\mathcal{A}) = \frac{\ker Q}{\text{im } Q} \ . \tag{7.21}$$

With definition (7.17) we are thus consistent with the literature (e.g. [Tom]) since in BRST quantization $H^*_Q(\mathcal{A})$ is conveniently used as the Hilbert space of (on-shell) physical states.
To define orthogonal projectors recall that $Q^2 = Q^{+2} = 0$ and that the decomposition above yields that the restrictions

$$\begin{aligned} Q|_{\text{Im } Q^+} &: \text{Im } Q^+ \xrightarrow{\cong} \text{Im } Q \\ Q^+|_{\text{Im } Q} &: \text{Im } Q \xrightarrow{\cong} \text{Im } Q^+ \end{aligned} \tag{7.22}$$

define isomorphisms between Im Q and Im Q^+. So the graded commutator H (on the A side given by the Laplace operator $\Delta_{\mathrm{End}(E)}$)

$$[Q, Q^+] = QQ^+ + (-1)^{|Q||Q^+|+1} Q^+Q = QQ^+ + Q^+Q =: H \tag{7.23}$$

defines an automorphism on $K^\perp$ whose inverse shall be denoted by H^{-1}. This allows to define

$$\begin{aligned} \pi_Q &:= QH^{-1}Q^+ \\ \pi_{Q^+} &:= Q^+H^{-1}Q \end{aligned} \tag{7.24}$$

as orthogonal projectors onto the space of spurious respectively unphysical states. With

$$P := 1 - (\pi_Q + \pi_{Q^+}) \tag{7.25}$$

we denote the orthogonal projector onto the space of physical states. In this setup a propagator is required to project out all states except the spurious ones, precisely speaking it shall that shall propagate spurious into unphysical states. Here such a propagator may be written as

$$U = -H^{-1}Q^+ \tag{7.26}$$

that allows to rewrite

$$\pi_Q = -QU \quad \text{and} \quad \pi_{Q^+} = -UQ \ . \tag{7.27}$$

7.2 A_∞-algebras and the vacua of cubic string field theory

Not only for working physicists, but also for mathematicians, these stated concepts may somehow be familiar in particular with regard to the usage of A_∞-algebras. This connection can be derived when using U to rewrite (7.25) as follows

$$P - 1 = QU + UQ \ , \tag{7.28}$$

that can be seen as the defining equation for

$$(K, Q_K) \underset{P}{\overset{\iota}{\rightleftarrows}} (\mathcal{A}, Q_{\mathcal{A}}, \cdot)$$

to be a *homotopy retract*. In that particular case this means the following:

- $(\mathcal{A}, Q_{\mathcal{A}}, \cdot)$ is a D.G.A. with differential $Q_{\mathcal{A}} \equiv Q$ and associative multiplication given by the string product $\cdot$. To remain consistent with the literature we denote $m_{\mathcal{A}}(u, v) := u \cdot v$.
- $(K = \bigoplus_{m\in\mathbb{Z}} (K \cap \mathcal{A}^m), Q_K)$ is a cochain complex where $K \subset \mathcal{A}$ as above denotes the space of physical states. Remark that $Q_K \equiv Q$ yielding $Q|_K \equiv 0$ per definition of K.

- $P : \mathcal{A} \to K$ is the previously defined projector onto the space of physical states K. The map ι denotes the inclusion $K \hookrightarrow \mathcal{A}$ and thus in particular is a chain map. We further need ι to be a *quasi-isomorphism*, that is it induces an isomorphism on cohomological level. This is true since

$$H^*_{Q_K}(K) = \frac{\ker Q_K}{\operatorname{im} Q_K} \overbrace{=}^{Q_K|_K=0} K \overbrace{\cong}^{(7.21)} H^*_{Q_{\mathcal{A}}}(\mathcal{A}) = \frac{\ker Q_{\mathcal{A}}}{\operatorname{im} Q_{\mathcal{A}}} . \tag{7.29}$$

- U (the propagator) is a degree -1 operator on $\mathcal{A}$.

- The identity $\iota \circ P - 1 = Q_{\mathcal{A}} U + U Q_{\mathcal{A}}$ holds.

Due to the work of J. Stasheff in [Sta] and thanks to S. Ma'u, who nicely explained this construction to us at the SFT V workshop 2011 in Hamburg, for such a setup the space of physical states can be equipped with an unfiltered A_∞-algebra structure

$$(K, \{m_n\}_{n\geq 1}) \tag{7.30}$$

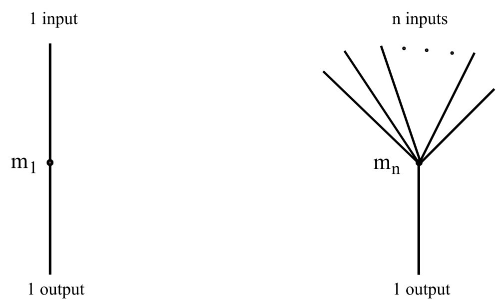
over R.
We shortly recap this construction amongst others by making use of the description of the A_∞-structures by using planar trees. Recall section 3.1.1 where unfiltered A_∞-algebra structures were introduced by equipping a graded vector space K (here we even do not have to be that general to use modules) over R with degree $2-n$ homomorphisms

$$m_n : K^{\otimes n} \to K \tag{7.31}$$

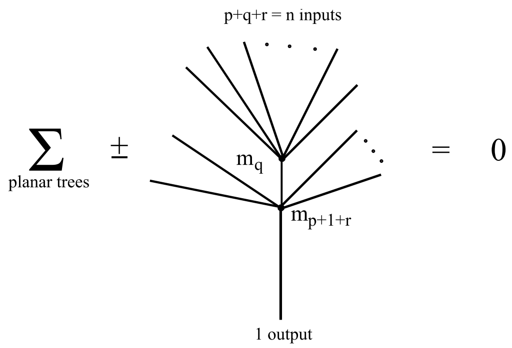
satisfying

$$\sum_{p+q+r=n} \pm\, m_{p+1+r}(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}) = 0 \quad \text{for all} \quad n \geq 1 . \tag{7.32}$$

Remark that there exist different conventions in the literature concerning the degree of m_n. In contrast to section 3.1.1, we use $2-n$ (cohomology convention since deg m_1=1) here, instead of $+1$, for the degree of m_n. Further we simplify things since we do not worry about sign issues that is, as in e.g. (7.32) we write $\pm$ instead of $(-1)^{p+q\cdot r}$.
To switch to the description by using planar trees, m_n is graphically illustrated as



and thus (7.32) may be visualized by



In order to further simplify writings, we define

$$\partial m_n := \pm\ m_n(m_1 \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \pm\ m_n(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes m_1) \pm\ m_1 \circ m_n\ , \quad (7.33)$$

that allows to rewrite (7.32) for particular n as follows

$\underline{n=1}$:

$$m_1 \circ m_1 = 0 \qquad (m_1 \text{ is a differential})$$
$$\Longleftrightarrow$$
$$\partial m_1 = 0$$

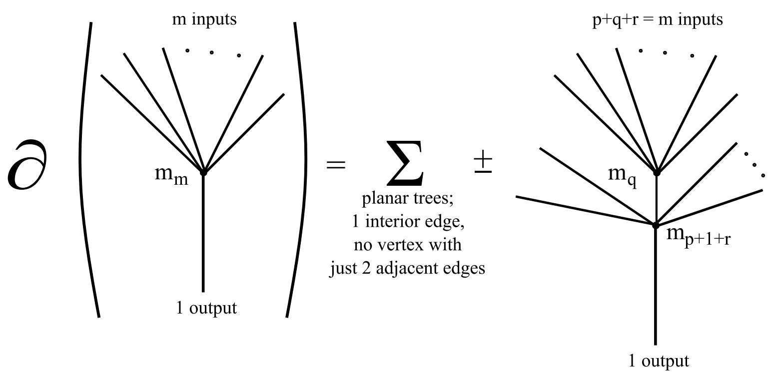
$\underline{n=2}$:

$$m_2(m_1, \mathrm{id}) + m_2(\mathrm{id}, m_1) - m_1 \circ m_2 = 0$$
$$(\text{Leibniz rule for product } m_2)$$
$$\Longleftrightarrow$$
$$\partial m_2 = 0$$

$\underline{n=3}$:

$$m_2(\mathrm{id}, m_2) - m_2(m_2, \mathrm{id}) =$$
$$m_3(m_1, \mathrm{id}, \mathrm{id}) + m_3(\mathrm{id}, m_1, \mathrm{id}) + m_3(\mathrm{id}, \mathrm{id}, m_1) + m_1 \circ m_3$$
$$(\text{Deviation of } m_2 \text{ of not being associative})$$
$$\Longleftrightarrow$$
$$\partial m_3 = m_2(\mathrm{id}, m_2) - m_2(m_2, \mathrm{id})$$

$\underline{n=m}$:



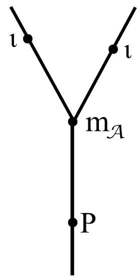
With that knowledge in mind we are prepared to write down the homomorphisms m_n for the A_∞-algebra $(K, \{m_n\}_{n\geq 1})$:
For $m_{n=1}$ we simply take the already given differential Q_K

$$m_1 = Q_K\ . \tag{7.34}$$

A product m_2 on K shall be defined as

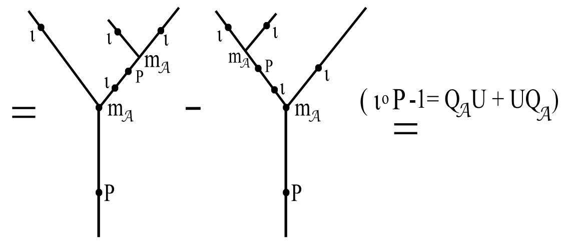
$$\begin{aligned} m_2 : K^{\otimes 2} &\to K \\ (u,v) &\mapsto P(m_{\mathcal{A}}(\iota(u), \iota(v))) \end{aligned} \tag{7.35}$$

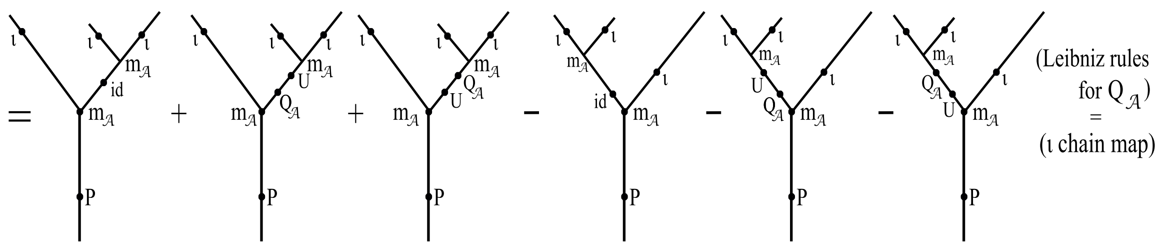
visualized by:

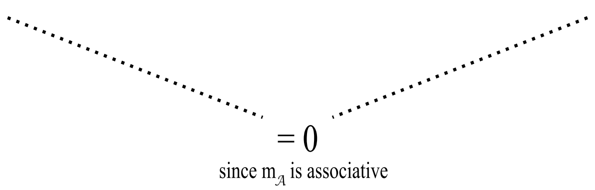


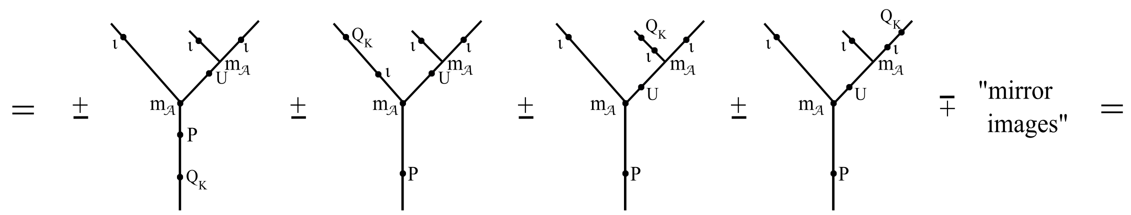
To derive an expression for m_3 we examine how the defined homomorphism m_2 deviates from being an associative multiplication:

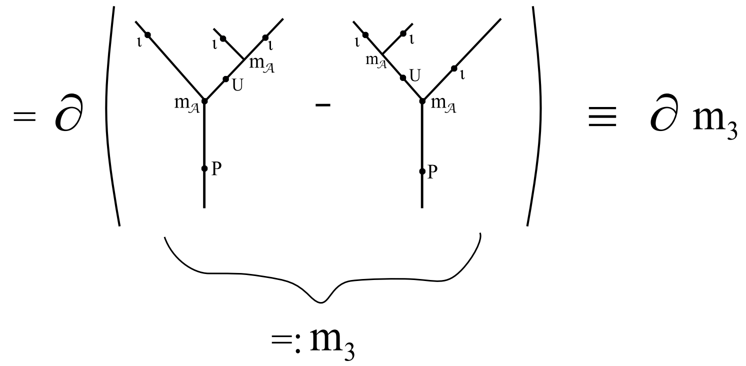
$$m_2(\mathrm{id}, m_2) - m_2(m_2, \mathrm{id}) = \tag{7.36}$$



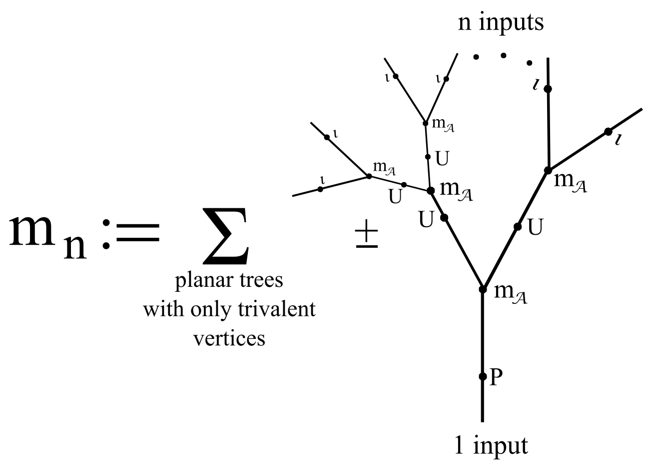








In that fashion we recursively define



and it can be shown that in such a case $(K, \{m_n\}_{n\geq 1})$ carries the structure of an unfiltered A_∞-algebra, called the *minimal model* in physicists language.
So where does the physics lie behind these planar trees and the present A_∞ description? In fact physicists interpret the maps

$$m_n : K^{\otimes n} \to K \tag{7.37}$$

as string products of physical states $u_1, ..., u_n \in K$, obeying the tree-level Feynman rules of the cubic open string field theory. For physicists the described planar tree picture thus displays the visualization of scattering processes by using Feynman diagrams. In that sense tree-level scattering amplitudes are defined as

$$\langle\langle u_1, ..., u_n \rangle\rangle^{(n)} := \langle u_1, m_{n-1}(u_2, ..., u_n)\rangle \tag{7.38}$$

When knowing $\langle\langle \cdot\cdot\cdot \rangle\rangle^{(n)}$, one defines a tree-level potential Ψ (also known as the *superpotential*) on the space of physical states of worldsheet degree one $K^1 = K \cap \mathcal{A}^1$.

This superpotential assigns $\psi \in K^1$ to the sum of all signed amplitudes of ψ for at least three leg processes, that is

$$\begin{aligned}
\Psi : K^1 &\to R \\
\psi &\mapsto \sum_{n\geq 3} \frac{1}{n}(-1)^{\frac{n(n-1)}{2}} \langle\langle \psi, ..., \psi\rangle\rangle^{(n)} \underbrace{\equiv}_{(7.38)} \\
&\equiv \sum_{n\geq 3} \frac{1}{n}(-1)^{\frac{n(n-1)}{2}} \langle \psi, m_{n-1}(\psi, ..., \psi)\rangle = \\
&= \sum_{n\geq 2} \frac{1}{n+1}(-1)^{\frac{n(n+1)}{2}} \langle \psi, m_n(\psi^{\otimes n})\rangle \ .
\end{aligned} \tag{7.39}$$

Meaning of Ψ for physicists:

In the standard cubic formulation of string field theory the relevant moduli space of vacua $\mathcal{M}$ is formed by solutions of the string field equations of motion

$$\mathcal{M} := \{\phi_0 \in \mathcal{A}^1 |\ Q\phi_0 + \frac{1}{2}[\phi_0, \phi_0] = 0\} \tag{7.40}$$

modulo certain gauge transformation. As nicely described in [Laz] this solution space can analogously be described when working with the data of the minimal model (which in contrast needs the gauge fixing $Q^+\phi = 0$)

$$(K, \{m_n\}_{n\geq 1}) \ . \tag{7.41}$$

Precisely speaking the author describes that the moduli space of solutions of the homotopy Maurer-Cartan equation

$$\mathcal{M}_\Psi := \{\phi_0 \in K^1 |\ \frac{\partial \Psi}{\partial \phi}|_{\phi=\phi_0} = 0\} \tag{7.42}$$

and $\mathcal{M}$ are *locally isomorphic* (meaning explained below)

$$\mathcal{M} \overset{\text{loc.}}{\cong} \mathcal{M}_\Psi \ . \tag{7.43}$$

This simplifies work in a way such that physicists, when facing concrete problems, are free to choose which approaches can be followed, either solving the equations of motion or searching critical points of Ψ, in order to describe the moduli space of vacua of cubic string field theory.

It remains to illustrate how to interpret the local isomorphism in (7.43).
As described above we construct the A_∞-algebra $(K, \{m_n\}_{n\geq 1})$ out of the given D.G.A. $(\mathcal{A}, \cdot, Q)$. Recall (1.21), namely $(\mathcal{A}, \cdot, Q)$ can in turn be considered as an A_∞-algebra

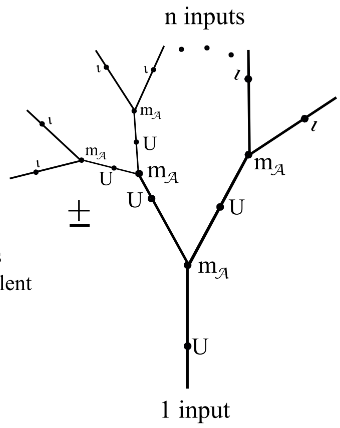
$$(\mathcal{A}, \{m'_n\}_{n\geq 1}) \tag{7.44}$$

with $m'_n = 0$ for $n \geq 3$. C. I. Lazaroiu showed that one can find an (unfiltered) A_∞-homomorphism

$$f = \{f_n\}_{n\geq 1} : (K, \{m_n\}_{n\geq 1}) \to (\mathcal{A}, \{m'_n\}_{n\geq 1}) \tag{7.45}$$

that in particular defines a weak homotopy equivalence (see Remark 3.1 (ii)) between them. Here f is defined as

$$\mathrm{f}_{\mathrm{n}} := \sum_{\substack{\text{planar trees}\\ \text{with only trivalent}\\ \text{vertices}}} \pm$$



for $n \geq 2$ that is quite similar to m_n as defined previously, however with the projector P replaced by the propagator U at the outgoing edge. For $n = 1$ it is defined by

$$f_1 = \iota : K \to \mathcal{A} \tag{7.46}$$

that clarifies that f is indeed a weak homotopy equivalence (meaning that f_1 is a quasi-isomorphism) since ι serves as a isomorphism between

$$K \cong H^*_{m_1'=Q}(\mathcal{A}) \equiv \frac{\text{ker } m_1'}{\text{im } m_1'} \tag{7.47}$$

as already described in (7.29).
Following this line of argumentation we get that $(K, \{m_n\}_{n\geq 1})$ and $(\mathcal{A}, \{m_n'\}_{n\geq 1})$ are even quasi-isomorphic as L_∞-algebras $(K, \{l_n\}_{n\geq 1})$ and $(\mathcal{A}, \{l_n'\}_{n\geq 1})$. Here we will not discuss the theory of L_∞-algebras, which naturally arise as symmetrization

$$m_n^{(')} \to l_n^{(')} \tag{7.48}$$

of A_∞-algebras, and refer the reader to Appendix A3 of [FOOO1] for details.
With this knowledge in mind we can use a Theorem of M. Kontsevich (section 4.4. of [Ko]) providing that the deformation functors, associating the corresponding moduli spaces $\mathcal{M}$ and $\mathcal{M}_\Psi$ to $(\mathcal{A}, \{l_n'\}_{n\geq 1})$ and $(K, \{l_n\}_{n\geq 1})$ respectively, are equivalent and thus denoting the local isomorphism as in (7.43).

Meaning of Ψ for mathematicians:

We come full circle by demonstrating that Ψ can additionally be helpful for mathematicians. Again we are interested in its critical points but now with regard to the detection of strict Maurer-Cartan solutions (see Definition 3.7). We follow the ideas of section 3.6.4. in [FOOO1].
Again we are working in the setup of $L \subset M$ being a special Lagrangian submanifold inside a Calabi-Yau threefold M. Assume it is equipped with a filtered A_∞-algebra structure

$$(C_L, m = \{m_k\}_{k\geq 0}) \tag{7.49}$$

where $C_L := H^*(L;\mathbb{Q}) \otimes \Lambda_{0,nov}$.
In order to define a potential function (of the form of the superpotential (7.39)) whose critical points are in one-to-one correspondence with strict Maurer-Cartan solutions we need to make use of two yet unproven conjectures.

(i) A pairing
$$\langle \cdot, \cdot \rangle : C_L \otimes C_L \to \Lambda_{0,nov} \tag{7.50}$$
is defined that satisfies
$$\langle \rho_0, m_k(\rho_1, ..., \rho_k)\rangle = (-1)^{(\deg\ \rho_k+1)\cdot(\deg\ \rho_0+...+\deg\ \rho_{k-1}+k)} \langle \rho_k, m_k(\rho_0, ..., \rho_{k-1})\rangle\ . \tag{7.51}$$

(ii) We can choose the homomorphisms m_k such that $\widehat{d}(e^b)$ is defined for all $b \in C_L^1$, that is the positive energy requirement $b \equiv 0$ mod $\Lambda^+_{0,nov}$ is redundant.

For a chosen basis $e_1, ..., e_n$ of $H^1(L;\mathbb{Q})$ and elements of the form $b = \sum_{i=1}^{n} x_i e_i \in C_L^1$, conjecture (ii) allows to define a potential
$$\begin{aligned} \Psi : \underbrace{\Lambda_{0,nov} \times ... \times \Lambda_{0,nov}}_{n} &\to \Lambda_{0,nov} \\ (x_1, ..., x_n) &\mapsto \sum_k \frac{1}{k+1}\langle b, m_k(b, ..., b)\rangle\ . \end{aligned} \tag{7.52}$$

Again asking about its critical points we get
$$\begin{aligned} \frac{\partial}{\partial x_j}\Psi(x_1, ..., x_n) = \sum_k \frac{1}{k+1}\langle e_j, m_k(b, ..., b)\rangle + \sum_k \frac{1}{k+1}\langle b, m_k(e_j, ..., b)\rangle + \cdots \\ \cdots + \sum_k \frac{1}{k+1}\langle b, m_k(b, ..., e_j)\rangle \overbrace{=}^{\text{(i)}} \\ = \sum_k \frac{1}{k+1}\langle e_j, m_k(b, ..., b)\rangle + \cdots \\ \cdots + (-1)^{\overbrace{(\deg\ e_j+1)}^{=\ 2\ =\ (\deg\ b+1)}(\cdots)} \sum_k \frac{1}{k+1}\langle e_j, m_k(b, ..., b)\rangle = \\ = \sum_k \langle e_j, m_k(b, ..., b)\rangle = \langle e_j, m(e^b)\rangle \end{aligned} \tag{7.53}$$

and thus for $b_0 = \sum_{i=1}^{n} x_{0,i} e_i$ we conclude
$$\nabla\Psi|_{\mathbf{x}_0=(x_{0,1},...,x_{0,n})} = 0 \quad \Leftrightarrow \quad \widehat{d}(e^{b_0}) \underbrace{=}_{(3.102)} e^{b_0} m(e^{b_0}) e^{b_0} = 0 \tag{7.54}$$

which finishes the observation that critical points of Ψ can be used as strict Maurer-Cartan solutions and vice versa.

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Erklärung

Hiermit versichere ich, Johannes Huster, diese Masterarbeit mit dem Titel *"A_∞-structures in Lagrangian Floer and String Field Theory"* selbstständig und nur unter Verwendung der im Literaturverzeichnis angegebenen Hilfsmittel und Quellen angefertigt zu haben.

München, October 11, 2011

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